Online Appendix for
"The Paradox of Toil at the Zero Lower Bound in a TANK Model"

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Abstract

In this Online Appendix, we first provide the detailed derivations of the model. Then, we describe further details on the paper's additional analyses and results.

Appendix A

This Appendix provides a detailed derivation of the New Keynesian Phillips equation and the IS equation reported in Eqs. (17a) and (17b), as well as Eqs. (33a), and (33b), respectively.

A.1 Derivation of the IS equation and interest rate rule

Let the variable with the subscript "L" refer to its steady-state value in the long-run and the lowercase variable denote the ratio of the deviation in the corresponding variable from its steady-state level to the steady-state output (e.g., $c_t^o = (C_t^o - C_L^o)/Y_L$ is the ratio of the deviation in the consumption of the optimizing households C_t^o from its steady-state C_L^o to the steady-state output Y_L). Based on the Keynes-Ramsey rule of the optimizing household reported in Eq. (3f), we can infer the following Keynes-Ramsey rule of the optimizing household with the linearized deviation form:

$$c_t^o = E_t c_{t+1}^o - \bar{\vartheta}^{-1}(i_t - E_t \pi_{t+1} - r_t^n), \tag{A1}$$

where $\pi_{t+1} = p_{t+1} - p_t$, $p_{t+1} = (P_{t+1} - P_L)/P_L$, $p_t = (P_t - P_L)/P_L$, and other parameters are the same as those defined in Subsection 3.1.

By using Eqs. (2), (3e), (5), (6d), (9), (13), (16a), (16b), $C_t = \varpi C_t^{RoT} + (1-\varpi)C_t^o$, $Y_t(j) = L_t(j)$ $D_t^{RoT} = D_t^o, V_t^{RoT} = 0$, and $V_t = (1-\varpi)V_t^o$, we can derive the following consumption decision between the two types of households with the linearized deviation form:

$$c_t^{RoT} = \varpi^{-1}(1-\varpi)\Upsilon c_t^o - \Gamma \tilde{\tau}_t^w, \tag{A2}$$

where $c_t^{RoT} = (C_t^{RoT} - C_L^{RoT})/Y_L$, $\Upsilon = \Omega_3^{-1}(\Omega_2 - \Omega_3)$, and other parameters are the same as those defined in Subsection 3.1.

By substituting $C_t = \varpi C_t^{RoT} + (1-\varpi)C_t^o$ into Eq. (16b), we obtain the economy's resource constraint with the linearized deviation form:

$$y_t = \varpi c_t^{RoT} + (1-\varpi)c_t^o, \tag{A3}$$

where $y_t = (Y_t - Y_L)/Y_L$. By combining Eqs. (A2) and (A3) together and inserting the resulting expression into Eq. (A1), we obtain the IS equation with the linearized deviation form reported in Eq. (17b):

$$y_t = E_t y_{t+1} - \sigma(i_t - E_t \pi_{t+1} - r_t^n) + \varpi\Gamma(E_t \tilde{\tau}_{t+1}^w - \tilde{\tau}_t^w), \tag{A4}$$

where $\sigma = (1-\varpi)\Lambda\bar{\vartheta}^{-1}$, $\Lambda = \Omega_2/\Omega_3$, and other parameters are the same as those defined in Subsection 3.1.

When we extend our original TANK model to consider a more general case where the RoT and optimizing households receive different amounts of nominal profits (i.e., $D_t^o \neq D_t^{RoT}$) and bear different amounts of the lump-sum taxes (i.e., $V_t^o \neq V_t^{RoT}$), the IS equation of the original TANK model reported in Eq. (A4) can then be modified as:

$$y_t = E_t y_{t+1} - \bar{\sigma}(i_t - E_t \pi_{t+1} - r_t^n) + \varpi\bar{\Gamma}(E_t \tilde{\tau}_{t+1}^w - \tilde{\tau}_t^w), \tag{A5}$$

Eq. (A5) is exactly the modified IS equation reported in Eq. (33b), and the parameters in Eq. (A5) are defined in Subsection 5.3.

Finally, from Eq. (15) we can derive the following interest rate rule:

$$i_t = \max(0, r_t^n + \phi_\pi \pi_t + \phi_y y_t)\,. \tag{A6}$$

A.2 Derivation of the New Keynesian Phillips curve

We follow Eggertsson (2011) to derive the New Keynesian Phillips curve in terms of the linearized deviation form. By combining Eqs. (3e), (6d), (9), and (16b) together with the production function $Y_t(j) = L_t(j)$ and inserting the resulting expression into Eq. (12), we have:

$$E_t \sum_{s=0}^{\infty} (\theta\beta)^s (\hat{p}_t^*(j) - \hat{p}_{t-1}) = E_t \sum_{s=0}^{\infty} (\theta\beta)^s (\hat{p}_{t+s} - \hat{p}_{t-1})$$
$$+ E_t \sum_{s=0}^{\infty} (\theta\beta)^s \frac{1}{1+\varepsilon\varphi} \left((\bar{\vartheta} + \varphi) y_{t+s} + (1-\tau_L^w)^{-1} \tilde{\tau}_{t+s}^w \right), \tag{A7}$$

where $\hat{p}_t^*(j) = (P_t^*(j) - P_L)/P_L$, $\hat{p}_{t-1} = (P_{t-1} - P_L)/P_L$ and $\hat{p}_{t+s} = (P_{t+s} - P_L)/P_L$. Given that the intermediate goods firm keeps its price unchanged with probability θ, the price index reported in Eq. (10) can then be rewritten as:

$$P_t = \left((1-\theta)(P_t^*(j))^{1-\varepsilon} + \theta P_{t-1}^{1-\varepsilon} \right)^{1/(1-\varepsilon)}. \tag{A8}$$

Then, based on Eq. (A8) and inflation $\Pi_t = P_t / P_{t-1}$, we can derive the following inflation with the linearized deviation form:

$$\pi_t = (1-\theta)(\hat{p}_t^*(j) - \hat{p}_{t-1})\,. \tag{A9}$$

Finally, by substituting Eq. (A9) into (A7), we obtain the NKPC with the linearized deviation form reported in Eq. (17a) (or Eq. (33a)):

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + \kappa\psi\chi\tilde{\tau}_t^w\,, \tag{A10}$$

where $\kappa = \bar{\kappa}(\bar{\vartheta} + \varphi)$, $\bar{\kappa} = (1-\theta)(1-\theta\beta)/\left(\theta(1+\varepsilon\varphi)\right)$, $\psi = (\bar{\vartheta} + \varphi)^{-1}$ and $\chi = (1-\tau_L^w)^{-1}$.

Appendix B

In this Appendix, we first briefly discuss the existence and uniqueness of the economy's stable equilibrium under two distinct regimes reported in Subsections 3.1 and 3.2. Then, we deal with Condition ZLB and Condition ZLBT that the ZLB interest rate policy is binding during the entire period in the presence of the adverse natural rate shock.

Following Woodford (2003), we consider a linear rational-expectations model of the following form:

$$E_t z_{t+1} = F z_t + f e_t\,, \tag{B1}$$

where z_t is a two-vector of non-predetermined (jump) endogenous state variables, e_t is a vector of temporary exogenous disturbances, and F and f are a two-by-two matrix and a two-vector of coefficients, respectively. Based on this linear rational-expectations model, by letting λ_1 and λ_2 be the two characteristic eigenvalues of the dynamic system, the rational-expectations equilibrium is *determinate* if and only if the matrix F has both eigenvalues outside the unit circle (i.e., the absolute values of the two eigenvalues are greater than one: $|\lambda_1|>1$ and $|\lambda_2|>1$). In line with Woodford (2003) and Eggertsson (2011), the condition for a unique bounded solution is satisfied if and only if either Case I or Case II holds:[1]

(i) Case I: $\det F>1$, $\det F-\operatorname{tr}F>-1$ and $\det F+\operatorname{tr}F>-1$;

(ii) Case II: $\det F-\operatorname{tr}F>-1$ and $\det F+\operatorname{tr}F<-1$.

B.1 Derivation of the unique bounded equilibrium under the interest rate rule regime

Based on the dynamic system under the interest rate rule regime reported in Eq. (19), we can derive that $\operatorname{tr}A=1+\beta^{-1}+\sigma\kappa\beta^{-1}+\sigma\phi_y>1$ and $\det A=\beta^{-1}\left(1+\sigma(\phi_y+\kappa\phi_\pi)\right)>1$, and thus the condition for Case II clearly does not apply. We now turn to consider the conditions for Case I. Equipped with $\phi_y>0$ and $\phi_\pi>1$, we can infer from $\operatorname{tr}A$ and $\det A$ that $\det A>1$, $\det A-\operatorname{tr}A>-1$, and $\det A+\operatorname{tr}A>-1$ hold, and hence the dynamic system under the interest rate rule regime achieves a unique bounded equilibrium.

B.2 Derivation of the unique bounded equilibrium under the ZLB interest rate regime

Based on the dynamic system under the ZLB interest rate regime reported in Eq. (22a), we can derive that $\operatorname{tr}\overline{A}=\mu^{-1}\left(1+\beta^{-1}(1+\sigma\kappa)\right)>1$ and $\det\overline{A}=\mu^{-2}\beta^{-1}>1$, and thus the condition for Case II clearly does not apply. We now turn to consider the conditions for Case I. First, we can infer from $\operatorname{tr}\overline{A}$ and $\det\overline{A}$ that both $\det\overline{A}>1$ and $\det\overline{A}+\operatorname{tr}\overline{A}>-1$ hold. Second, to satisfy that $\det\overline{A}-\operatorname{tr}\overline{A}>-1$, we need to impose the restriction that $(1-\mu)(1-\beta\mu)-\kappa\mu\sigma>0$. This restriction is Condition UBE reported in Eq. (23).

B.3 Derivation of Condition ZLB and Condition ZLBT in the presence of the adverse natural rate shock

This Appendix provides a detailed proof of Condition ZLB in Subsection 3.2.3 and Condition ZLBT in Section 4. To be more specific, Condition ZLB and Condition ZLBT are imposed to satisfy the following requirement: In the presence of an adverse natural rate shock r_S^n, the shock is sufficiently large so that the nominal interest rate endogenously falls below zero and thus forces the monetary authority to implement the ZLB interest rate policy.

In line with Eggertsson (2011, p.105), by using Eqs. (17a) and (17b) together with the interest rate rule $i_t=r_S^n+\phi_\pi\pi_t+\phi_y y_t$ ($t\in[0,T^e-1]$), we can derive the dynamic system with the interest rate rule in the presence of the negative natural rate shock:

$$E_t\begin{bmatrix}\pi_{t+1}\\ y_{t+1}\end{bmatrix}=\hat{A}\begin{bmatrix}\pi_t\\ y_t\end{bmatrix}+\hat{a}\tilde{\tau}^w, \qquad \text{(B2)}$$

where

[1] See Woodford (2003, p. 670) for a detailed derivation of the proof.

$$\hat{A} = \begin{bmatrix} (\beta\mu)^{-1} & -\kappa(\beta\mu)^{-1} \\ \sigma(\phi_\pi - \beta^{-1})\mu^{-1} & (1+\sigma\beta^{-1}\kappa + \sigma\phi_y)\mu^{-1} \end{bmatrix}, \quad \hat{a} = \begin{bmatrix} -(\beta\mu)^{-1}\kappa\psi\chi \\ \left(\varpi\Gamma(1-\mu) + \sigma\beta^{-1}\kappa\psi\chi\right)\mu^{-1} \end{bmatrix}. \tag{B3}$$

According to Eq. (B3), we can derive the results: $\mathrm{tr}\,\hat{A} = \mu^{-1}\left(1+\beta^{-1}+\sigma(\kappa\beta^{-1}+\phi_y)\right) > 1$ and $\det\hat{A} = \mu^{-2}\beta^{-1}\left(1+\sigma(\phi_y+\kappa\phi_\pi)\right) > 1$, and thus the condition for Case II does not apply. We now turn to deal with the restrictions that satisfy the conditions for Case I. First, by using $\mathrm{tr}\,\hat{A}$ and $\det\hat{A}$, we can obtain that $\det\hat{A} > 1$ and $\det\hat{A} + \mathrm{tr}\,\hat{A} > -1$. Second, in order to satisfy the condition $\det\hat{A} - \mathrm{tr}\,\hat{A} > -1$, we impose the restriction $(\phi_\pi - 1) + (1-\beta)\kappa^{-1}\phi_y > -\kappa^{-1}(1-\mu)\left((1-\beta\mu)/\sigma + (\kappa+\phi_y)\right)$. Then, based on this restriction, we can infer that the short-run dynamic system under the interest rate rule is associated with the feature of global instability. In addition, by inserting $E_t\pi_{t+1} = \mu\pi_S + (1-\mu)\pi_L$, $E_t y_{t+1} = \mu y_S + (1-\mu)y_L$, and $y_L = \pi_L = 0$ into Eqs. (17a) and (17b), the economy's equilibrium under the interest rate rule regime can be summarized by the following two linear expressions: the short-run aggregate demand (SAD) equation and the short-run aggregate supply (SAS) equation:

$$SAD: \quad y_S = -\frac{\sigma}{1-\mu+\sigma\phi_y}(\phi_\pi - \mu)\pi_S - \frac{\varpi\Gamma(1-\mu)}{1-\mu+\sigma\phi_\pi}\tilde{\tau}^w, \tag{B4}$$

$$SAS: \quad y_S = \frac{1-\beta\mu}{\kappa}\pi_S - \psi\chi\tilde{\tau}^w. \tag{B5}$$

By using Eqs. (B4) and (B5), we can easily derive a short-run equilibrium of output and inflation under the interest rate rule regime as follows:

$$y_S = -\left(\frac{\varpi\Gamma(1-\mu)(1-\mu\beta) + \sigma(\phi_\pi - \mu)\kappa\psi\chi}{(1-\beta\mu)(1-\mu+\sigma\phi_y) + \kappa\sigma(\phi_\pi - \mu))}\right)\tilde{\tau}^w, \tag{B6}$$

$$\pi_S = -\left(\frac{\kappa\varpi\Gamma(1-\mu) - (1-\mu+\sigma\phi_y)\kappa\psi\chi}{(1-\mu+\sigma\phi_y)(1-\beta\mu) + \kappa\sigma(\phi_\pi - \mu)}\right)\tilde{\tau}^w. \tag{B7}$$

Then, by inserting Eqs. (B6) and (B7) into the interest rate rule $i_S = r_S^n + \phi_\pi\pi_S + \phi_y y_S$, we can infer that the short-run equilibrium of the interest rate can be described as follows:

$$i_S = r_S^n + \Gamma_{\tau^w}\tilde{\tau}^w, \tag{B8}$$

where

$$\Gamma_{\tau^w} = -\frac{\left(\phi_\pi\kappa\mu + \phi_y(1-\mu\beta)\right)\varpi\Gamma(1-\mu) - \left((1-\mu)\kappa\psi\phi_\pi + \sigma\mu\kappa\psi\phi_y\right)\chi}{(1-\mu+\sigma\phi_y)(1-\beta\mu) + \kappa\sigma(\phi_\pi - \mu)}. \tag{B9}$$

Eq. (B8) reveals the following two results. First, when the fiscal authority does not implement a supportive payroll tax cut policy ($\tilde{\tau}^w = 0$), the monetary authority is forced to implement a ZLB interest rate policy if $i_S = r_S^n < 0$ occurs. This is Condition ZLB reported in Eq. (26). Second, when the fiscal authority implements an additional payroll tax cut policy ($\tilde{\tau}^w < 0$), the monetary authority is forced to implement the ZLB interest rate if $i_S = r_S^n + \Gamma_{\tau^w} \tilde{\tau}^w < 0$ occurs. As indicated in Condition ZLBT reported in Eq. (28), this implies that the natural rate shock is sufficiently large such that a payroll tax cut policy implemented by the fiscal authority cannot fully offset the adverse effect arising from the natural rate shock and thus the ZLB is binding.

Appendix C

Armed with the restrictions $\alpha^{RoT} = 1$ and $\varsigma^{RoT} = 0$ in Section 4, this Appendix provides a detailed derivation of the decomposition of the representative RoT household's disposable income. By using Eq. (5), the representative RoT household disposable income is defined as its net wage income plus profit income minus lump-sum taxes:

$$C_t^{RoT} = \underbrace{(1-\tau_t^w)\int_0^1 \frac{W_t(j)}{P_t} L_t^{RoT}(j)dj}_{\text{net wage income}} + \underbrace{\alpha^{RoT}\int_0^1 \frac{D_t(j)}{P_t}dj}_{\text{profit income}} - \underbrace{\varsigma^{RoT} V_t}_{\text{RoT lump-sum tax}} . \tag{C1}$$

Equipped with $\alpha^{RoT} = 1$ and $\varsigma^{RoT} = 0$, from (C1) we can then infer the following expression:

$$C_t^{RoT} = \underbrace{(1-\tau_t^w)\int_0^1 \frac{W_t(j)}{P_t} L_t^{RoT}(j)dj}_{\text{net wage income}} + \underbrace{\int_0^1 \frac{D_t(j)}{P_t}dj}_{\text{profit income}} . \tag{C2}$$

Based on Eq. (C2), changes in the disposable income of the representative RoT household can then be expressed as the sum of the changes in net wage income and changes in profit income in the following linearized deviation form:

$$c_t^{RoT} = \underbrace{(1-\tau_L^w)\zeta^{-1}\int_0^1 \hat{w}_t^r(j)dj + (1-\tau_L^w)\zeta^{-1}\int_0^1 l_t^{RoT}(j)dj - \zeta^{-1}\tilde{\tau}_t^w}_{\text{changes in net wage income}} + \underbrace{[(1-\zeta^{-1})y_t - \zeta^{-1}\int_0^1 \hat{w}_t^r(j)dj]}_{\text{changes in profit income}} . \tag{C3}$$

Eq. (C3) reveals two important implications regarding changes in the disposable income of the RoT household. First, changes in net wage income are composed of three items. The first item $(1-\tau_L^w)\zeta^{-1}\int_0^1 \hat{w}_t^r(j)dj$ reflects changes in the gross wage (i.e., the pre-tax wage). The second item $(1-\tau_L^w)\zeta^{-1}\int_0^1 l_t^{RoT}(j)dj$ reflects changes in the RoT employment. The third item $-\zeta^{-1}\tilde{\tau}_t^w$ denotes changes in the payroll tax rate. Second, changes in profit income are composed of two items. The first item $(1-\zeta^{-1})y_t$ reflects changes in the demand for the intermediate inputs used in the final good production. The second item $-\zeta^{-1}\int_0^1 \hat{w}_t^r(j)dj$ reflects changes in the intermediate good firms' marginal cost, which in turn is related to changes in the gross wage.

By substituting $\int_0^1 \hat{w}_t^r(j)dj = (\overline{\vartheta}+\varphi)y_t + (1-\tau_L^w)^{-1}\tilde{\tau}_t^w$ and $\int_0^1 l_t^{RoT}(j)dj = ((\varphi+\overline{\vartheta})/\varphi)y_t - (\overline{\vartheta}/\varphi)c_t^{RoT}$ into Eq. (C3), we have:

$$c_t^{RoT} = \underbrace{(1-\tau_L^w)\zeta^{-1}[(\bar{\vartheta}+\varphi)y_t + (1-\tau_L^w)^{-1}\tilde{\tau}_t^w] + (1-\tau_L^w)\zeta^{-1}\{[(\varphi+\bar{\vartheta})/\varphi]y_t - (\bar{\vartheta}/\varphi)c_t^{RoT}\} - \zeta^{-1}\tilde{\tau}_t^w}_{\text{changes in net wage income}} + \underbrace{\{(1-\zeta^{-1})y_t - \zeta^{-1}[(\bar{\vartheta}+\varphi)y_t + (1-\tau_L^w)^{-1}\tilde{\tau}_t^w]\}}_{\text{changes in profit income}} \tag{C4}$$

Rearranging Eq. (C4) yields:

$$c_t^{RoT} = \frac{(\Omega_2 - \Omega_1)}{\Omega_2} y_t - \frac{(\tau_L^w)^{-1}(1-\tau_L^w)^{-1}}{\Omega_2}\tilde{\tau}_t^w, \tag{C5}$$

where $\Omega_1 = 1 + \bar{\vartheta} + \varphi > 0$ and $\Omega_2 = (\tau_L^w)^{-1}[\zeta + \bar{\vartheta}(1-\tau_L^w)/\varphi] > 0$. Eq. (C5) clearly indicates that, with a given output, a payroll tax cut tends to raise the RoT household's disposable income. The reason for this result is quite clear by referring to Eq. (C4). As indicated in (C4), the item $(1-\tau_L^w)\zeta^{-1}[(1-\tau_L^w)^{-1}\tilde{\tau}_t^w]$ for changes in net wage income is lower than the item $-\zeta^{-1}[(1-\tau_L^w)^{-1}\tilde{\tau}_t^w]$ for changes in profit income. With the additional positive item $-\zeta^{-1}\tilde{\tau}_t^w$ for changes in net wage income, we can then assert that a payroll tax cut leads to a rise in the RoT household's disposable income.

We further explore how the downward shift in the SAD curve is related to changes in the RoT households' disposable income following a reduction in the payroll tax rate. Substituting $c_t^o = E_t c_{t+1}^o - \bar{\vartheta}^{-1}\{i_t - E_t\pi_{t+1} - r_t^n\}$, $c_t^{RoT} = \frac{(\Omega_2 - \Omega_1)}{\Omega_2} y_t - \frac{(\tau_L^w)^{-1}(1-\tau_L^w)^{-1}}{\Omega_2}\tilde{\tau}_t^w$, $E_t y_{t+1} = \mu y_S + (1-\mu) y_L$, $E_t\pi_{t+1} = \mu\pi_S + (1-\mu)\pi_L$, $E_t c_{t+1}^o = \mu c_S^o + (1-\mu)c_L^o$, $E_t\tilde{\tau}_{t+1}^w = \mu\tilde{\tau}_S^w + (1-\mu)\tilde{\tau}_L^w$, $i_t = 0$, $r_t^n = r_S^n$, $y_L = \pi_L = c_L^o = 0$, and $\tilde{\tau}_L^w = 0$ into the resource constraint $y_t = \varpi c_t^{RoT} + (1-\varpi)c_t^o$, we have:

$$y_S = \underbrace{\frac{\varpi(\Omega_2 - \Omega_1)}{\Omega_2} y_S - \varpi\frac{(\tau_L^w)^{-1}(1-\tau_L^w)^{-1}}{\Omega_2}\tilde{\tau}_S^w}_{\text{changes in RoT households' disposable income}} + \underbrace{\frac{(1-\varpi)\bar{\vartheta}^{-1}}{(1-\mu)}\{\mu\pi_S + r_S^n\}}_{\text{changes in optimizing households' disposable income}}. \tag{C6}$$

Eq. (C6) clearly shows that changes in the short-run equilibrium output can be decomposed into two components. The first is changes in the RoT households' disposable income. The second is changes in the optimizing households' disposable income arising from the intertemporal substitution effect.

Eq. (C6) can be rearranged as Eq. (24a) in the main text: $y_S = \sigma\frac{\mu}{1-\mu}\pi_S + \frac{\sigma}{1-\mu}r_S^n - \varpi\Gamma\tilde{\tau}^w$. From Eq. (24a), we can infer that the downward shift in the SAD line in association with a decline in the payroll tax rate is:

$$\left.\frac{\partial\pi_S}{\partial(-\tilde{\tau}^w)}\right|_{SAD} = -\frac{(1-\mu)\varpi\Gamma}{\sigma\mu}. \tag{C7}$$

The intuition behind Eq. (C7) is straightforward. The *downward* shift in the SAD curve is characterized by a given output level. As indicated in (C5), a payroll tax cut tends to raise the RoT households' disposable income. Then, to satisfy an unchanged output, the optimizing households' disposable income should fall in response, which can be achieved by a reduction in the inflation rate through the intertemporal substitution effect. As a result, in response to an unchanged output, a payroll tax cut raises the RoT households' disposable income and hence leads the SAD curve to shift downward.

Appendix D

This Appendix provides a detailed derivation of the IS equation and the law of motion of real government bonds reported in Eqs. (40b) and (40c), respectively. Due to the fact that the New Keynesian equation does not change regardless of whether the fiscal authority issues government bonds or balances its budget by the lump-sum tax, the New Keynesian equation reported in Eq. (40a) is the same as Eq. (17a).

D.1 Derivation of the law of motion for real government bonds

By using Eqs. (3e), (6d), (9), (37), (39a), (39c), $C_t = \varpi C_t^{RoT} + (1-\varpi)C_t^o$, $Y_t(j) = L_t(j)$, and $B_L / P_L = 0$, we can derive the following law of motion for real government bonds with the linearized deviation form:

$$b_{t+1} = (1+\rho-\delta_1)b_t - (\zeta)^{-1}\chi\tilde{\tau}_t^w - \tau_L^w(\zeta)^{-1}(1+\vartheta+\varphi)y_t . \quad \text{(D1)}$$

where $b_t = (B_t / P_{t-1} - B_L / P_L)/Y_L$, $b_{t+1} = (B_{t+1} / P_t - B_L / P_L)/Y_L$, and $\rho = r_L^n$.

D.2 Derivation of the IS equation

By using Eqs. (2), (3e), (5), (6d), (9), (16a), (16b), (13), (38a), (39b), (39c), $C_t = \varpi C_t^{RoT} + (1-\varpi)C_t^o$, $Y_t(j) = L_t(j)$, $D_t^{RoT} = \alpha^{RoT} D_t$ and $D_t^o = \alpha^o D_t$ we can derive the following consumption decision between the two types of households with the linearized deviation form:

$$c_t^{RoT} = \varpi^{-1}(1-\varpi)\ddot{\Upsilon}c_t^o - \ddot{\Gamma}\tilde{\tau}_t^w - \ddot{\Theta}b_t , \quad \text{(D2)}$$

where $c_t^{RoT} = (C_t^{RoT} - C_L^{RoT})/Y_L$, $\tilde{\tau}_t^w = \tau_t^w - \tau_L^w$, $\ddot{\Omega}_1 = \left((1-\alpha^{RoT})\zeta - (1-\alpha^{RoT} - \tau_L^w)(1+\vartheta+\varphi)\right)\big/\tau_L^w$, $\Omega_2 = \left(\zeta + (1-\tau_L^w)\vartheta / \varphi\right)/\tau_L^w$, $\ddot{\Omega}_3 = \varpi\ddot{\Omega}_1 + (1-\varpi)\Omega_2$, $\ddot{\Gamma} = \alpha^{RoT}\left(\tau_L^w(1-\tau_L^w)\ddot{\Omega}_3\right)^{-1}$, $\ddot{\Upsilon} = \ddot{\Omega}_3^{-1}(\Omega_2 - \ddot{\Omega}_3)$, and $\ddot{\Theta} = \varsigma^{RoT}\delta_1\varpi\zeta(\tau_L^w\ddot{\Omega}_3)^{-1}$.

By substituting $C_t = \varpi C_t^{RoT} + (1-\varpi)C_t^o$ into Eq. (16b), we obtain the economy's resource constraint with the linearized deviation form:

$$y_t = \varpi c_t^{RoT} + (1-\varpi)c_t^o , \quad \text{(D3)}$$

where $y_t = (Y_t - Y_L)/Y_L$. By combining Eqs. (D2) and (D3) together and inserting the resulting expression into Eq. (A1), we obtain the IS equation with the linearized deviation form:

$$y_t = E_t y_{t+1} - \ddot{\sigma}(i_t - E_t\pi_{t+1} - r_t^n) + \varpi\ddot{\Gamma}(E_t\tilde{\tau}_{t+1}^w - \tilde{\tau}_t^w) + \ddot{\Theta}(E_t b_{t+1} - b_t) , \quad \text{(D4)}$$

where $\ddot{\sigma} = (1-\varpi)\ddot{\Lambda}\vartheta^{-1}$ and $\ddot{\Lambda} = \Omega_2 / \ddot{\Omega}_3$. Furthermore, based on the TANK model reported in Section 6, the analytical analysis we consider is based on the assumption that the nominal profits of both types of households are equal (i.e., $D_t^o = D_t^{RoT} = D_t$ and thus $\alpha^{RoT} = 1$) and, by substituting

$\alpha^{RoT} = 1$ into Eq. (D4), we obtain the IS equation with the linearized deviation form reported in Eq. (40b):

$$y_t = E_t y_{t+1} - \sigma(i_t - E_t \pi_{t+1} - r_t^n) + \varpi \Gamma (E_t \tilde{\tau}_{t+1}^w - \tilde{\tau}_t^w) + \Theta(E_t b_{t+1} - b_t) \ , \qquad \text{(D5)}$$

where $\Omega_1 = 1 + \overline{\vartheta} + \varphi$, $\Omega_2 = (\zeta + (1 - \tau_L^w)\overline{\vartheta} / \varphi) / \tau_L^w$, $\Omega_3 = \varpi\Omega_1 + (1 - \varpi)\Omega_2$, $\Lambda = \Omega_2 / \Omega_3$, $\sigma = (1 - \varpi)\Lambda\overline{\vartheta}^{-1}$, $\Gamma = (\tau_L^w(1 - \tau_L^w))^{-1} / \Omega_3$, and $\Theta = \varsigma^{RoT}\delta_1\varpi\zeta(\tau_L^w\Omega_3)^{-1}$.

References

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