Optimal Taxation and Implementation in the Life Cycle with Human Capital Investment†

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Abstract

This paper studies the capital, labor and human capital wedges and the tax implementation in a lifecycle model with risky human capital when consumption is indistinguishable from education expenses. The planner faces asymmetric information regarding agents' exogenous abilities and endogenous human capital. Agents can thus deviate in two ways: by misreporting their ability and by mis-investing in their human capital. The complexity in the possible deviation strategies and difficulties in characterizing them make the problem so complex, but this paper analytically characterizes the distortions. Distortions to constrained efficient allocations are characterized by a capital wedge that is positive over life cycle and a labor wedge that is negative early and positive later in the life cycle. These wedges serve as mechanisms to eliminate the distortion to consumption due to its indistinguishability from education expenditure. We construct a simple tax system of linear capital and linear labor income tax rates to implement the constrained efficient allocations in a decentralized economy. Simulation results suggest that the average capital wedge is positive in all working periods, with progressive capital wedges against contemporaneous skill types, and the average labor wedge is negative in early and positive in later periods, with hump-shaped labor wedges against contemporaneous skill types and nonzero labor wedges at the top and the bottom of the skill distribution.

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1. Introduction

Human capital plays an important role in public's lives. People receive education when young and accumulate human capital through learning, training, education, and development over the life cycle. A wide range of goods and services have components of consumption and human capital investment, and it is difficult to distinguish consumption activities from education purposes. As Lazear (1977) pointed out, education is simply normal consumption and, like all other normal goods, an increase in income will produce an increase in the schooling purchase.[1] According to Bovenberg and Jacobs (2005), books, computers and travelling costs are difficult to verify, because individuals may misrepresent expenditure for consumption purposes as investment in education. Beside buying education related goods, people may hire tutors for private lessons, or may take supplementary classes at "cram schools".[2] Alongside classes for language skills and course work for private classes, people may pay private trainers to teach lessons for extracurricular activities.[3] These private lessons are costly, but it is not easy to differentiate the cost from other expenses. Such difficulty has been recognized in the policy debate on how to design the tax system in order to foster human capital accumulation.[4] As Grochulski and Piskorski (2010) put it, the center of this measurement problem lies in the fact that, in reality, there is a human capital investment component in normal consumption expenditure and a consumption value in the human capital investment activity such as education and training.[5] Indeed, as early as in 1961, Schultz (1961) was aware of the difficulty in distinguishing education from consumption expenditure.[6]

Human capital investment is affected by tax policy, as labor taxes deter human capital investment by capturing part of the return to human capital and capital taxes influence the choice of investment between physical capital and human capital investment. Moreover, human capital investment affects the tax base and is a major element of the income distribution. Though diverse across countries, policies that

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1 See also Weisbrid (1962), Heckman (1976), Trostel (1993, p.333), and Davies et al. (2000) for more discussion.

2 A cram school, called *Coaching Centers* in Australia and India, *Juku* in Japan, *Buxiban* in China and Taiwan, *Nachhilfe* in Germany, and *Hagwon* in Korea, is a private, fee-paying school. It offers classes for language skills or tests like BCT for Chinese language proficiency and IELTS for English language proficiency (e.g., Chou, 2015). It also offers test-training classes for college entrance exams or tests like SAT and GRE (e.g., Entrich, 2014).

3 Examples of extracurriculum activities are performing art lessons for musical devices (like piano, cello, and violin), sports (swimming, skating, and skiing) and dance (ballet, disco, and Latin), sculpture arts, and painting and drawing, among others. Moreover, many organizations like YMCA (https://www.ymca.net/) and Yamaha Music School (http://www.yamaha.com/ymes/musicschools/home.asp) offer classes for extra-curriculum activities.

4 For the policy debate, readers are referred to the reference and discussion in a memorandum to the President's Advisory Panel on Federal Tax Reform (2005) on tax treatment of investment in human capital.

5 According to Grochulski and Piskorski (2010, pp.909-910), agents use a large variety of goods, services, and non-market activities as vehicles for human capital investment as well as for consumption. It is difficult to measure the relative "loadings" of human capital investment and pure consumption embedded in a particular good or service.

6 "*We can think of three classes of expenditures: expenditures that satisfy consumer preferences ... expenditures that enhance capabilities... and expenditures that have both effects. Most relevant activities clearly are... partly consumption and partly (human capital) investment, which is why the task of identifying each component is so formidable and why the measurement of capital formation by expenditures is less useful for human investment.*" T.W. Schultz (1961, p.8).

affect human capital accumulation influence workers' skill distribution, which is a crucial feature in optimal income taxation models. Such an interaction between human capital and the tax system calls for an investigation of optimal income tax policies in a model with endogenous human capital over the life cycle. The vast majority of the literature on optimal income taxes assumes that productivity is exogenous in place of being the result of human capital decisions made throughout life.

In this paper, we study a dynamic Mirrlees model, in which agents accumulate privately observed human capital through education expenditure over the life cycle, but private consumption expenses may be disguised as expenses for education purposes. Recently, Stantcheva (2017) has explored a dynamic Mirrlees model with *observable* human capital investment over the life cycle. Our model extends Stantcheva (2017) to one with *unobservable and observable* human capital investment over the life cycle. This extension is not trivial, because agents can now deviate in two ways: by misreporting their ability and by mis-investing in their human capital. Mis-investing in human capital is not directly observed by the planner, and has persistent effects: an agent who has invested the wrong amount of human capital in the past will face different trade-offs today, and different incentives to report truthfully. It is precisely this complexity in possible deviation strategies and the difficulty in characterizing them that make the problem complex. Even so, we analytically solve the problem and characterize the distortions of the constrained efficient allocations in the planner's problem. Moreover, we construct a tax system to implement the constrained efficient allocations in the decentralized economy. Earlier, in a dynamic Mirrlees model with *unobservable* human capital investment only in the *initial period*, Grochulski and Piskorski (2010) constructed a tax system to implement the constrained efficient allocation in the decentralized economy. Our tax system is simpler and adds value to Grochulski and Piskorski (2010). Two main findings are as follows.

First, the distortions in the resulting constrained efficient allocations are characterized by a positive capital (or intertemporal) wedge, and a labor (or intratemporal) wedge that is negative early and positive later in the life cycle. These wedges are caused by *distortions to consumption* due to indistinguishable consumption from education expenditure.

Stantcheva (2017) and Grochulski and Piskorski (2010) obtained positive capital wedges due to skill shocks, which is an *insurance effect* against future risks.[7] Our paper finds a new mechanism, as consumption and education expenses are indistinguishable. As a result, agents may increase consumption by reducing unobservable human capital investment (hereafter HCI), dubbed the *HCI effect* for simplicity. A higher-skill shock today exerts two HCI effects: one is reducing education expenses for consumption today and the other is reducing education expenses for consumption tomorrow. While today's HCI effect enhances the positive capital wedge from the insurance effect, tomorrow's HCI effect counteracts the positive capital wedge from the insurance effect. The net capital wedge is in general positive, unless tomorrow's HCI effect is so strong that completely offsets the sum of today's HCI effect and the

[7] Models with exogenous skills (cf. Diamond and Mirrlees, 1978; Golosov et al., 2003) also obtained positive capital wedges due to the insurance effect.

insurance effect. In particular, in the terminal period, with neither the insurance effect nor tomorrow's HCI effect, there is only today's HCI effect, so the capital wedge is larger than the otherwise positive capital wedge arising from the insurance effect.

Moreover, our negative labor wedge early in the life cycle is different from a positive labor wedge in Stantcheva (2017) and a positive labor wedge for low-skilled agents as well as a negative labor wedge for high-skilled agents in Grochulski and Piskorski (2010). The standard Mirrlees model with exogenous skills obtained a positive labor wedge for the redistribution purpose (cf. Golosov et al., 2006), but a positive labor wedge in these two papers is to prevent the high-skilled from reducing labor effort through pretending to be low-skilled, labelled the *shirk-preventing effect*. However, with indistinguishable consumption and education expenses in our model, a negative labor wedge induces agents to work more today, so more consumption from under-investing in human capital is less attractive, called the *skill-fostering effect*. Intuitively, the deviation strategies involve shirking, which is combined with under-investing in human capital. The deviators are over-skilled relative to the truth-tellers, who provide the same low effective labor supply. The deviators have a stronger preference for leisure and a weaker preference for consumption than the truth-tellers. Thus, it is not worthwhile for the deviators to under-invest in human capital for more consumption. A marginal subsidy to labor income makes it optimal to offer the effective labor supply and invest in human capital according to their true types, thus a skill-fostering effect. In early periods, the skill-fostering effect dominates the shirk-preventing effect, so the labor wedge is negative. In later periods, the investment in human capital decreases and the skill-fostering effect phases out, so the labor wedge is positive.

Second, we implement the constrained efficient allocation in terms of linear labor and capital taxes. Our implementation is different from that in Grochulski and Piskorski (2010). These authors used non-linear labor income taxes to restrict agents' labor to the constrained efficient level, in order to ensure that agents do not jointly deviate their labor and savings from the constrained efficient allocation. However, this is possible, only if there is a deferred capital tax; if otherwise, their two linear capital-tax-adjusted Euler equations associated with both truth-telling and shirking may be inconsistent with each other. By contrast, our model does not use a deferred capital tax. We do not restrict agents' labor in our decentralized economy to the constrained efficient level as did in Grochulski and Piskorski (2010). We only use a mild condition to assure agents' after-tax labor income in consistency with the income in the constrained efficient allocation. With different shadow prices of this mild condition for different skill types and with linear labor income taxes, our two linear capital-tax-adjusted Euler equations associated with truth-telling and shirking are consistent with each other. It is easier to implement a linear labor income tax than a deferred capital tax.

Finally, we calibrate our model to the US data and illustrate the optimal tax policy. Simulation results suggest that the average capital wedge is positive over life cycle. The average capital wedge is decomposed into a standard positive insurance effect, a positive today's HCI effect, and a negative tomorrow's HCI effect. As today's HCI effect dominates tomorrow's HCI effect, the average capital wedge in our model is higher than the model without private human capital investment. Moreover, the scatter plot indicates that the standard insurance effect is positive and regressive against contemporaneous skill types, while today's HCI effect is positive and progressive and tomorrow's HCI effect is negative and diminishing against contemporary skill types. As today's HCI effect being 10 times as large as the insurance effect and tomorrow's HCI effect, the capital wedge is positive and progressive against contemporary skill types.

Unlike the standard positive average labor wedge with only a positive shirking-preventing effect in otherwise identical models without private human capital investment (e.g., Stantcheva, 2017) and without any human capital investment (e.g., Farhi and Werning, 2013), our simulation results suggest that the average labor wedge is negative early in the life cycle due to a negative skill-fostering effect dominating the standard positive shirking-preventing effect. Moreover, the scatter plot indicates that the labor wedge is hump-shaped against contemporaneous skill types, like that in the new dynamic public finance literature (e.g., Golosov et al., 2006); yet, as the negative skill-fostering effect quantitatively dominates the positive shirking-preventing effect, the labor wedge is negative at the top and the bottom of the productivity distribution, thus different from the standard zero-tax result. Note that our nonzero-tax result comes from private human capital investment, different from the nonzero-tax result in Farhi and Werning (2013), which arises from a moving support with the top and bottom bounds of the productivity being functions of the previous period's productivity.

In addition, our history-dependent optimal tax policy results in a large welfare gain when comparing with a laissez-faire economy without taxes. If our history-dependent tax policy is too complicated to be feasible, a simple history-independent non-linear tax policy adopted by Heathcote et al. (2017) gives a very close welfare gain. Lastly, in order to check the envelope condition in the relaxed planning problem is satisfied ex-post, we will quantitatively verify that the allocation solved by the relaxed planning problem in this paper is indeed incentive compatible.

1.1 Related literature

There is a long-standing literature that emphasized endogenous skill acquisition (Becker, 1964; Ben-Porath, 1967; Heckman, 1976). In particular, the literature posited that human capital is accumulated over the life cycle, underscoring the need for a life-cycle model (Cunha and Heckman, 2007). A body of empirical work documents that the return to human capital investment, and thus the earning, is risky (e.g. Palacios-Huerta, 2003; Meghir and Pistaferri, 2004; and Storesletten et al., 2004). Our model incorporates these facts by assuming that agents invest in human capital throughout the course of their life cycle and the accumulation of human capital is subject to shocks.

There is a growing literature named new dynamic public finance, which extends the optimal taxation pioneered by Mirrlees (1971) to a dynamic setting. As opposed to the Ramsey approach, which specifies ex ante the instruments available to the government, the Mirrlees approach adopted here considers an unrestricted direct revelation mechanism.[8] Yet, the literature typically considers exogenously evolving

[8] For the Ramsey approach, readers are referred to Judd (1985), Chamley (1986) and, more recently, Chen and Lu (2013) and Krueger and Ludwig (2013). For the Mirrlees approach, readers are referred to Golosov et al. (2003), Kocherlakota (2005), Albanesi and Sleet (2006), Golosov et al. 2006, Werning (2007), Farhi et al. (2012), and Farhi

abilities, thus abstracting from endogenous skill acquisition. Our paper contributes to this literature by taking endogenous skills into account, which change over time based on human capital investment.

Our paper is closely related to the dynamic Mirrlees literature that studied optimal income taxation in models with monetary investment in human capital.[9] As mentioned earlier, our paper is most related to the models of Stantcheva (2017) and Grochulski and Piskorski (2010).

First, in Stantcheva (2017), agents accumulate human capital over the life cycle through *observable* education expenses.[10] As a result, in addition to positive optimal labor wedge for *redistribution* purposes and positive optimal capital wedge for *insurance* purposes, she uncovered optimal education subsidies to observable education expenses. Thus, subsidies to education not only prevent agents from shirking but also offset capital and labor income *tax-induced distortions to learning*.[11] By contrast, in our model, agents accumulate human capital through both *unobservable* and observable education expenses. As a result, our positive capital wedges arise not only from *insurance* purposes but also from the *HCI effect* that emerges from distortions to consumption due to its inseparability from education expenses. Moreover, our negative labor wedge, and thus an implicit subsidy to labor, early in the life cycle arises because the positive effect from working according to their types (the *shirk-preventing effect*) is dominated by the negative effect from investing in human capital due to inseparability of consumption from education expenses (the *skill-fostering effect*). Therefore, our tax policy is to offset distortions due to consumption being indistinguishable from education expenditure.[12]

In an earlier version, Stantcheva (2014, Section 7) has extended the Stantcheva (2017) model to one with unobservable human capital, wherein consumption is not differentiable from human capital expenses. She derived the condition for the adjusted labor wedge and the modified inverse Euler equation for the capital wedge, but the signs of the labor wedge and the capital wedge are not determined. Our model has three differences. Firstly, while skill shocks arrive before agents invest in human capital in our model, skill shocks come after human capital investment is made in Stantcheva (2014). As the skill shock arrives in the period before the human capital investment is made, the skill shock first affects agent's investment in human capital and then affects agents' stock of human capital with a one-period lag. The difference in the timing of skill shocks enables our model to separate the *skill-fostering effect* from the

and Werning (2013), among others.

[9] Another investment in human capital is through time. See, e.g., Kapička (2006, 2015), da Costa and Masestri (2007), Maldonado (2008), Boháček and Kapička (2008), Anderberg (2009, Kapička and Neira (2015), and Stantcheva (2015).

[10] Like Stantcheva (2017), Findeisen and Sach (2016) also studied a dynamic Mirrlees model with costly education investment, but their focus is on one-shot uncertain college education investment before the work life starts. Makris and Pavan (2019) studied a human capital model, but the evolution of human capital comes from learning-by-doing, as a by-product of working, not from education investment. Our investment in human capital model is through resources and over the life cycle, and thus different from Findeisen and Sach (2016) and Makris and Pavan (2019).

[11] In a static model earlier, Bovenberg and Jacobs (2005) also found positive optimal labor taxes for redistribution and education subsidies to eliminate the adverse impact of redistributive taxes on observable education expenses.

[12] Like Stantcheva (2017), we also consider a subsidy to observable education expenses. Yet, the subsidy to observable education expenses cannot fully resolve the problem of misreporting unobservable education expenses.

shirk-preventing effect, which work at different timings with one-period differences. Secondly, our model determines the signs of labor and capital wedges. We analytically obtain a negative labor wedge in early periods of the life cycle, at least in the first period, when the negative *skill-fostering effect* on the labor wedge dominates the positive *shirk-preventing effect*. Moreover, we obtain a modified inverse Euler equation, which analytically separates today's positive *HCI effect* from tomorrow's negative *HCI effect* on the capital wedge, so as to make assure when the capital wedge is larger or smaller than the capital wedge in the case with only observable human capital. Thirdly, we implement the labor and capital wedge in terms of linear labor and capital income taxes in a decentralized economy, which was not analyzed in the case with unobservable human capital in Stantcheva (2014).

Next, in Grochulski and Piskorski (2010), human capital can be increased by making *unobservable* education expenses at the *initial* period and is subject to stochastic depreciation shocks later in the life cycle.[13] As a result, these authors found a positive labor wedge for a low-skilled type and a negative labor wedge for a high-skilled type, and also a positive capital wedge, which emerged from the *insurance effect* due to stochastic depreciation shocks.[14] Moreover, they implemented the constrained efficient allocation in terms of linear capital taxes and non-linear labor taxes in the decentralized economy, with the requirement of deferred capital taxes so their linear capital tax-adjusted Euler equations associated with truth-telling and shirking strategies are consistent with each other. Skill shocks come after human capital investment is made in Grochulski and Piskorski (2010). Our model is different. Firstly, skill shocks arrive before agents invest in human capital in our model. Moreover, agents accumulate human capital by making *unobservable* education expenses through their *life cycle* in our model. As a result, for all skill types, the labor wedge is negative early and positive later in the life cycle; and the capital wedge is positive, but is from not only *insurance effects* but also *HCI effects*. Moreover, with a minor income condition, we implement the constrained efficient allocation in terms of linear labor and capital taxes without requiring a deferred capital income tax. This is possible, because the shadow price of the income condition makes room for the constrained efficient allocation to satisfy the two linear capital tax-adjusted Euler equations associated with shirking and truth-telling agents.

Moreover, Kapička (2015) and Kapička and Neira (2019) have studied Mirrlees income taxes in models with observable and unobservable human capital investment through the life cycle. As a result of risky human capital investment, Kapička (2015) found that the optimal marginal income taxes decrease with age, and Kapička and Neira (2019) discovered that optimal tax policies balance redistribution across agents, insurance against human capital shocks, and incentives to learn and work. With incentives to learn and work, an increase of labor supply in one period induces changes in human capital investments in other periods, which in turn affects the disutility of working in other periods in these two papers. As unobservable human capital formation makes preferences over labor supply non-separable across age,

[13] In their model, initial endowment may be consumed, or invested in human capital, so agents become high-skilled. Later, bad shocks arrive stochastically, and if got hit by shocks, a high-skilled type becomes low-skilled forever.

[14] They also demonstrated the premium of human relative to physical capital that arises from unobservable human capital investment. For simplicity, our paper does not analyze the premium of human relative to physical capital.

optimal marginal income taxes depend on not only whether labor is complementary or substitutable across age, but also how incentives between learning and work are balanced. Different from observable consumption and unobservable learning time for human capital formation in these two papers, our paper considers unobservable consumption and observable and unobservable education expenditure for human capital formation. As a result of indistinguishable consumption from education expenditure, our labor wedge is negative in early ages and positive in later ages, thus implying optimal implicit labor income taxes to increase with age. Kapička and Neira (2019) also studied the capital wedge, but their positive capital wedge comes from the moral hazard problem in order to elicit higher learning effort today, not from the private information problem. By contrast, our positive capital wedge comes from not only the insurance problem (*insurance effects*) but also from the tradeoff between unobservable consumption and unobservable education expenditure (*HCI effects*).[15]

The feature of unobservable consumption in our model is reminiscent of Allen (1985) and Cole and Kocherlakota (2001). Allen (1985) analyzed whether the optimal long-term contract is better than a series of unrelated short-term contracts, when agents can borrow secretly. Cole and Kocherlakota (2001) characterized whether efficient consumption allocation can be decentralized through a competitive asset market, wherein agents can store asset secretly. As agents can borrow or save secretly, agents' consumption is unobservable in these two existing papers. In contrast, our unobservable consumption emerges, because agents' consumption is indistinguishable from education expenditure. In particular, our paper investigates whether unobservable consumption affects the design of the optimal income tax policy on capital and labor, which was not studied by these two papers.[16]

To use the tax policy to implement the constrained efficient allocation in a decentralized economy, Kocherlakota (2005) implemented the constrained efficient allocation by restricting to linear capital and nonlinear labor income taxes with zero capital tax rate on *average*, but the capital tax rate depends in general on the history of labor income reports. Albanesi and Sleet (2006) proposed non-linear and non-separable optimal taxes that depend on current wealth and income only, but it applies only in a setting with shocks that are i.i.d.; i.e., it ruled out persistence of productivity over time. Following the tax structure in Kocherlakota (2005) with history-dependent, non-linear labor taxes, Grochulski and Piskorski (2010) found that, by allowing the deferred taxation of capital income, linear capital income taxes and non-linear labor income taxes can implement the constrained efficient allocations in a decentralized economy. By contrast, with an income tax condition, we use linear capital and linear labor income taxes to implement the constrained efficient allocation in a decentralized economy.

[15] We should also refer to the paper by Bovenberg and Jacobs (2010) that introduced verifiable and non-verifiable human capital investment in a two-period model. However, their human capital investment is in terms of time effort, so there is no indistinguishable consumption from education expenditure.

[16] Shourideh (2014) studied savings and bequest wedges in a dynamic model with unobservable consumption. His model is totally different from our model. While his model abstracts from human capital, our model considers observable and unobservable human capital.

Finally, Diamond (1980) and Saez (2002) obtained a negative labor wedge at the bottom of the income distribution in models with an extensive margin of the labor supply. They found that subsidies to the working poor are optimal, because the labor force participation effect dominates the incentive effect of higher income earners. Our negative labor wedge is different, as it is a subsidy to all workers. Moreover, the reason is different, due to indistinguishable consumption from education expenditure.

We organize this paper as follows. In Section 2, we present a two-period model. In Section 3, we study the social planner's problem and characterize the sign of the capital, labor and net human capital wedges of the constrained efficient allocation. Section 4 extends the model to T periods. In Section 5, we provide a tax system to implement the constrained efficient allocation in a market economy. In Section 6, we offer numerical analysis. Finally, concluding remarks are offered in Section 7.

2. An Illustrative Example: A Two-period Model

In order to introduce the main ideas and understand the main results in the simplest possible way, we start with a two-period model.[17]

2.1 The environment

The economy consists of a continuum of agents who live for two periods. An agent obtains utility from consumption and disutility from working, with a utility function represented by:

$$u(c_1)-\phi(l_1)+\beta\left[u(c_2)-\phi(l_2)\right],$$

where $0<\beta<1$ is the discount factor, c_t is consumption and l_t is work effort in period t. An agent provides at most $\bar{l}>0$ work effort in a period. We assume that $u(c)$ is continuously differentiable, strictly increasing and concave, and satisfies the Inada condition, and $\phi(l)$ is continuously differentiable, strictly increasing and convex, and satisfies $\phi(0)=0$, $lim_{l\to 0}\phi'(l)=0$, and $lim_{l\to\bar{l}}\phi'(l)=\infty$.

At the initial period $t=1$, an agent's disposable income may be consumed, spent on education to accumulate human capital h_2, or saved to form physical capital k_2 in the next period. There are two kinds of education expenses, verifiable x_t and non-verifiable y_t. The human capital technology is $\psi(x_t,y_t)$, which is strictly increasing and strictly concave in x and y; that is, $\psi_{xx}<0<\psi_x$ and $\psi_{yy}<0<\psi_y$. Acquisition of human capital is subject to skill shocks. Given an initial human capital h_1 and its depreciation rate δ_h, the human capital evolves as follows.

Assumption 1. The evolution of human capital in the next period is:

$$h_2=(1-\delta_h)h_1+\psi(x_1,y_1)+\theta,$$

where θ is a skill shock over a fixed support $\Theta=[\underline{\theta},\overline{\theta}]$.[18]

[17] Later, we will extend the model to T periods and show that the results derived in a two-period model continue to hold. A T-period model is much more complicated, as choices and shocks an agent has made and experienced in the past will affect human capital investment.

[18] Skill shocks are introduced in order to be in line with the setup in the existing literature. For simplicity, time

A skill shock θ is an innate ability, which is private information, and will be referred to as an agent's type. To yield a positive capital wedge, it is not necessary for the ability shock to change over time. In this illustrative two-period model, the ability shock can be treated as constant,[19] and hence, the positive capital wedge arises from the private human capital investment, not for the insurance purpose. When the model is extended to T periods (cf. Section 4 below), our model introduces a stochastic ability shock, so a positive capital wedge arises also from the insurance purpose obtained in the existing literature.

Two remarks are in order. First, in our model, the innate ability shock arrives in the first period before human capital investment is made, which is different from Grochulski and Piskorski (2010) and Stantcheva (2017), wherein skill shocks come after human capital investment is made. If skill shocks come after human capital investment is made, there is the shirking preventing effect, which gives a positive labor wedge. With unobservable human capital investment, there is also the skill fostering effect, which yields a negative labor wedge. However, these two effects are intertwined and thus, it is difficult to analyze the net effect. Next, if the skill shock is not separately from human capital investment, the effect of skill shocks is intricate when the model is extended to T periods (cf. Section 4 below), as then the skill shock θ_t changes over time and the whole shock history $\theta^t = (\theta_1, \ldots, \theta_t)$ will directly affect human capital investment in a complicated way. To simplify the problem, we assume that the human capital technology is separable in education expenses and skill shocks. Note that, in Stantcheva (2017), the wage rate, which plays a role like effective human capital in our model, is a function of human capital and skill shocks. The wage rate gives the Hicksian coefficient of complementarity between human capital and skill types, with human capital increasing the wage inequality if the complementarity coefficient is positive, but decreasing the wage inequality if otherwise. With the evolution of human capital over time, if we follow the setup in Stantcheva (2017), unobservable human capital investment gives rise to intertwining effects, creating an unnecessary complication that possibly won't achieve what we hope in analysis. A separable human capital technology gives a zero-complementarity coefficient between human capital investment and skill types, so human capital investment has a neutral effect on the wage inequality, which simplifies the analysis.[20] In particular, under the separability assumption, the shirking preventing effect is analytically separated from the skill fostering effect. With the timing that the innate ability shock arrives before human capital investment is made, the skill fostering effect dominates in early periods, which ensures that the labor wedge is unambiguously negative at least in the first period.

The probability of the realization of a skill shock θ is $\pi(\theta)$, with $0 \leq \pi(\theta) \leq 1$ for each $\theta \in \Theta$. We assume that the realization of θ is identically and independently distributed (i.i.d.) for each agent.

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subscripts to skill shocks are dropped in the two-period model here. Later, when the model is extended to T periods, time subscript t will be added.

[19] Although skill shocks are time varying, agent's types in period 2 do not affect the economy in a simple two-period model, for human capital investment is not needed in the terminal period. Thus, agents only need to report their types in period 1. In a T-period model latter, the history of types affects the allocation in later periods.

[20] If we allow for the complementarity between human capital investment and skill types, the result is qualitatively unchanged, though the positive net human capital wedge (meaning subsidy) is enhance.

Suppose that the law of large numbers applies; then, $\pi(\theta)$ also indicates the fraction of agents whose skill shock is θ. For simplicity, agents are assumed to endow with the same human capital level h_1 when born, while agents with larger skill shocks have advantages to acquire human capital more effectively than those with smaller shocks. An agent with human capital h_t and work effort l_t supplies $z_t = l_t h_t$ units of effective labor.

In each period, the representative firm combines aggregate physical capital K_t and aggregate effective labor Z_t to produce final goods using the technology $F(K_t, Z_t)$. The technology is neoclassical, which satisfies constant returns to scale and is strictly increasing and concave in K_t and Z_t. The physical capital depreciates at the rate of δ_k.

In our environment, agents' skill shocks θ, work effort in the second period l_2 and non-verifiable education expenses in the first period y_1 are private information. Thus, human capital in the second period h_2 is private information. Moreover, individual consumption c_1 and non-verifiable education expenses y_1 in the first period are not distinguishable, so c_1 is also private information. By contrast, initial human capital h_1, individual physical capital k_t and individual effective labor z_t, $t = 1,2$, verifiable education expenses x_1, and consumption in the second period c_2 are publicly observable. Note that work effort l_1 is inferable from initial human capital and individual effective labor in the first period z_1, and is thus observable. Although consumption c_1 and non-verifiable education expenses y_1 are not publicly observable, their sum $c_1 + y_1$ is observable, since it is inferable from the budget constraint. In the special case when there are no non-verifiable education expenses, consumption c_1 is observable and our model reduces to the same model as Stantcheva (2017) except that skill shocks arrive before agents choose human capital investment.

2.2 Resource feasibility

In order to maximize the social welfare, the social planner would have chosen to equally allocate consumption for agents of all types. However, as work effort and education expenses are private information, such allocation is not incentive-compatible, because this would encourage higher-skill agents to reduce work effort and education expenses. To avoid this situation, according to the Revelation Principle, feasible allocations need to be incentive compatible. Below, we start by defining the resource feasible allocation, so we have notations ready for establishing the incentive compatible constraint.

In the first period, agents report their types $\theta \in \Theta$. The social planner allocates the resource to agents according to the reported type. Following the notation used by Farhi and Werning (2013), we use $\sigma(\theta)$ to denote an agent's reporting strategy, specifying a reported type σ conditional on the true type θ. Some notations are in order. If a is observable, $a(\theta)$ denotes the allocation to true type θ. If a is unobservable, then $a^\sigma(\theta)$ denotes the allocation to true type θ whose reported type is σ. When $\sigma = \theta$, the agent truthfully reports the type and thus, $a(\theta)$ denotes the allocation to the truth-telling agent.

An allocation $A \equiv (c, y, x, l, h, z, k, Z, K)$ specifies consumption $c \equiv \{c_1^\sigma(\theta), c_2(\theta)\}$, non-verifiable education expenses $y \equiv \{y_1^\sigma(\theta)\}$, verifiable education expenses $x = \{x_1(\theta)\}$, work effort $l \equiv$

$\{l_1(\theta), l_2^\sigma(\theta)\}$, human capital $h \equiv \{h_1, h_2^\sigma(\theta)\}$, effective labor $z \equiv \{z_1(\theta), z_2(\theta)\}$, where $z_t = h_t l_t$, and physical capital $k \equiv \{k_1, k_2(\theta)\}$ for all $\sigma, \theta \in \Theta$, aggregate effective labor $Z \equiv \{Z_1, Z_2\}$ and physical capital $K \equiv \{K_1, K_2\}$, given h_1 and $k_1 = K_1$.

Given K_1, h_1, G_1 and G_2, an allocation A is ***resource feasible*** if

$$\int_\Theta \pi(\theta)[c_1(\theta) + y_1(\theta) + x_1(\theta)]d\theta + K_2 \le F(K_1, Z_1) + (1-\delta_k)K_1 - G_1, \tag{1a}$$

$$\int_\Theta \pi(\theta)c_2(\theta)d\theta \le F(K_2, Z_2) + (1-\delta_k)K_2 - G_2, \tag{1b}$$

$$c_1^\sigma(\theta) + y_1^\sigma(\theta) = c_1(\sigma) + y_1(\sigma), \tag{1c}$$

$$h_2^\sigma(\theta) = (1-\delta_h)h_1 + \psi\left(x_1(\sigma), y_1^\sigma(\theta)\right) + \theta, \tag{1d}$$

where $K_2 = \int \pi(\theta)k_2(\theta)d\theta$, $Z_t = \int \pi(\theta)z_t(\theta)d\theta$ and G_t is the government expenditure in $t = 1,2$.

While (1a) and (1b) are the resource constraints and (1d) is the evolution of human capital, as noted earlier, (1c) is needed because the sum of an agent's consumption and non-verifiable education expenses in period 1 is observable. Thus, to avoid being caught, an agent with reporting strategy $\sigma(\theta)$ needs to maintain the reported sum $c_1^\sigma(\theta) + y_1^\sigma(\theta)$ equal to the sum $c_1(\sigma) + y_1(\sigma)$ reported by true type σ.

2.3 The agent's problem

As consumption is indistinguishable from education expenditure, the agent with a reporting strategy $\sigma(\theta)$ may reallocate expenses between consumption and education, as long as the sum of these expenses is consistent with the reported type. Given the allocation $\{c_1 + y_1, x_1, c_2, k_2, z_1, z_2\}$ that the social planner assigns for truth-telling agents, an agent θ with a reporting type σ will choose the allocation $\{c_1, y_1, h_2\}$ to solve the following problem:

$$\max_{c_1, y_1, h_2} u(c_1) - \phi\left(\frac{z_1(\sigma)}{h_1}\right) + \beta\left[u(c_2(\sigma)) - \phi\left(\frac{z_2(\sigma)}{h_2}\right)\right], \tag{2}$$

s.t. $c_1 + y_1 = c_1(\sigma) + y_1(\sigma)$ and $h_2 = (1-\delta_h)h_1 + \psi(x_1(\sigma), y_1) + \theta$, with h_1 given.

Although the allocations $\{c_1, y_1, h_2\}$ are not observed by the public, the social planner knows that agents with a reporting strategy $\sigma(\theta)$ would privately choose allocation $\{c_1^\sigma(\theta), y_1^\sigma(\theta), \ h_2^\sigma(\theta)\}$ by solving the above problem. To simplify the notation, we denote $\phi_h\left(\frac{z_t}{h_t}\right) \equiv -\phi'\left(\frac{z_t}{h_t}\right)\frac{z_t}{(h_t)^2} < 0$. Then, the following proposition can be proved directly from the first-order conditions of the above problem.[21]

Proposition 1. *When non-verifiable education expenses exist, the resource feasible allocation A satisfies*

[21] All the proofs for the propositions and lemmas in this paper are relegated to the Appendix. The proof of Proposition 1 is in Appendix A.1.

$$u'(c_1^\sigma(\theta)) = -\beta\phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\psi_y(x_1(\sigma), y_1^\sigma(\theta)). \tag{3}$$

Furthermore, $\frac{\partial y_1^\sigma(\theta)}{\partial\theta} < 0$ *and* $\frac{\partial h_2^\sigma(\theta)}{\partial\theta} > 0.$

The proposition says that, an agent with a reporting strategy $\sigma(\theta)$ spends on non-verifiable education y_1 until the decrease in the marginal utility of consumption today equal to the increase in the discounted marginal utility of leisure tomorrow (or the decrease in the discounted marginal disutility of labor) resulting from higher human capital. The result $\frac{\partial y_1^\sigma(\theta)}{\partial\theta} < 0$ means that, given the same reporting type σ, agents who underreport their type (i.e. $\theta > \sigma$) would choose to invest less of non-verifiable education expenses than truth-telling agents (i.e., $y_1^\sigma(\theta) < y_1^\sigma(\sigma)$). The intuition behind $\frac{\partial y_1^\sigma(\theta)}{\partial\theta} < 0$ and $\frac{\partial h_2^\sigma(\theta)}{\partial\theta} > 0$ are explained as follows. Suppose there are two types of agents, wherein one is type θ and the other has a higher type $\theta' > \theta$. Suppose that both high-type θ' and low-type θ agents underreport their types as σ; that is, $\sigma < \theta < \theta'$. This implies that high-type agents θ' can save more than low-type agents θ by cutting down non-verifiable education expenses (i.e., $y_1^\sigma(\theta') < y_1^\sigma(\theta) < y_1^\sigma(\sigma)$). Note that, when underreporting their types, even though high-type agents cut more non-verifiable education expenses than low-type agents, they do not save too much from cutting non-verifiable education expenses to lose their advantages in human capital, and thus, the relationship $h_2^\sigma(\theta') > h_2^\sigma(\theta) > h_2^\sigma(\sigma)$ remains the same.

Note that if $y_1(\theta) = 0$, our model reduces to one with only verifiable education expenses $x_1(\theta)$. Thus, unless $\frac{\partial h_2(\theta)}{\partial\theta} > 0$, Proposition 1 does not apply.

2.4 Incentive compatibility

Now, we establish the incentive compatible constraint. Let the lifetime utility of an agent with reporting strategy $\sigma(\theta)$ be denoted by

$$W^\sigma(\theta) \equiv u\left(c_1^\sigma(\theta)\right) - \phi\left(l_1(\sigma)\right) + \beta\left[u\left(c_2(\sigma)\right) - \phi\left(l_2^\sigma(\theta)\right)\right]. \tag{4a}$$

where $c_1^\sigma(\theta)$ must satisfy Proposition 1, when there are non-verifiable education expenses. Then, an allocation A is ***incentive-compatible*** if

$$W(\theta) \geq W^\sigma(\theta), \forall\sigma, \theta \in \Theta. \tag{4b}$$

In setting up the social planner's problem, we go along with the method proposed for dynamic Mirrlees models by Farhi and Werning (2013), Kapička (2013) and Stantcheva (2017). The procedure goes through the following steps to make a tractable recursive formulation. First, a relaxed problem is written based on the first-order approach, which replaces the full set of incentive compatibility constraints by the envelope condition. Next, when the 2-period model is extended to T periods (see Section 4 below), this relaxed program will be turned into a recursive dynamic programming problem through a suitable definition of state variables.

The envelope condition is derived as follows. Incentive compatibility (IC) constraints in (4a)-(4b) imply that, for all θ, the incentive compatibility constraint is as follows.

$$W(\theta)=\max_{\sigma\in\Theta} W^{\sigma}(\theta)=\max_{\sigma\in\Theta} u\left(c_1(\sigma)+y_1(\sigma)-y_1^{\sigma}(\theta)\right)-\phi\left(\tfrac{z_1(\sigma)}{h_1}\right)+\beta\left[u\left(c_2(\sigma)\right)-\left(\tfrac{z_2(\sigma)}{h_2^{\sigma}(\theta)}\right)\right]. \tag{4c}$$

If we take the derivative with respect to (true) skill shocks, there are two direct effects on unobserved variables, namely, unobserved education expenses y_1 and the human capital h_2, and indirect effects on the allocation through the report. By the first-order conditions of the agent, all indirect effects are jointly zero and only the two direct effects remain. This leads to the agent's envelope condition as follows.

$$\dot{W}(\theta)=-u'\left(c_1(\theta)\right)\frac{\partial y_1(\theta)}{\partial\theta}+\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{\left(h_2(\theta)\right)^2}\frac{\partial h_2(\theta)}{\partial\theta}, \tag{4d}$$

where $\dot{W}(\theta)\equiv\frac{\partial W(\theta)}{\partial\theta}$ and $\frac{\partial h_2(\theta)}{\partial\theta}=\psi_y\left(x_1(\theta),y_1(\theta)\right)\frac{\partial y_1(\theta)}{\partial\theta}+1$. From now on, a variable with a dot on its top is used to denote the derivative of the variable with respect to a shock θ.

The envelope condition uncovers how a promised utility changes with types at incentive-compatible allocations. In order to encourage agents to tell the truth, an incentive compatible allocation must prevent agents from getting the benefit of misreporting. The first term in (4d) is the static utility gain from reducing non-verifiable education expenses for consumption today, while the second term is the dynamic utility gain of leisure tomorrow from the benefit of higher shirking abilities.[22] It has been shown that the envelope condition is a necessary condition for incentive compatibility (Milgrom and Segal, 2002).

3. The Planning Problem in the Two-period Model

We proceed to envisage the social planner's dynamic programming problem. By comparing the second-best allocation in the social planner's problem to the allocation in the decentralization problem, we can understand the distortion in the second-best allocation relative to the laissez-faire allocation.

The social planner chooses allocations that maximize the following utilitarian social welfare:[23]

$$Max \int_{\Theta} W(\theta)\pi(\theta)d\theta,$$

subject to resource constraints (1a)-(1b) and incentive compatibility constraints (4a)-(4b).

The following definition describes the second-best allocation.

[22] As seen from Proposition 1, with skill shocks, higher-skill types need not spend non-verifiable education expenses too much in order to yield higher skills in the future, which is beneficial for a shirking ability. Thus, their best misreporting strategy is to reduce non-verifiable education expenses a little bit today for more consumption today and more leisure tomorrow.

[23] See Diamond (1998) and Tuomala (1990) concerning how the choice of the welfare function affects optimal taxes in a static framework. For more general social welfares, readers are referred to Saez and Stantcheva (2016) as to how the tax policy is reformed under generalized social marginal weights, which is beyond the scope of this paper.

Definition 1. An allocation A is ***(utilitarian) constrained efficient*** if it maximizes the welfare of the utilitarian social planner in the class of all feasible incentive-compatible allocations.

The incentive-compatible allocation will be referred to as the constrained efficient allocation. As will be seen, the constrained efficient allocation does not satisfy the standard consumption Euler equation. This leaves a room for the benevolent government to impose the optimal wedges in order to offset distortions in the resulting the constrained efficient allocation.

3.1 The relaxed planning problem

The relaxed planning problem replaces the IC constraints (4a)-(4b) by the envelope condition (4d).[24] Let λ_t be the shadow price of the resource constraint in period t and $\mu(\theta)$ be the co-state variable associated with the envelope condition $\dot{W}(\theta)$. The Hamiltonian of the relaxed planning problem is relegated to the Appendix A.2. The boundary conditions of the Hamiltonian are

$$\mu(\underline{\theta}) = \lim_{\theta \to \underline{\theta}} \mu(\theta) = 0 \quad \text{and} \quad \mu(\overline{\theta}) = \lim_{\theta \to \overline{\theta}} \mu(\theta) = 0. \tag{5a}$$

We have derived the first-order conditions with respect to $c_1(\theta), z_1(\theta), z_2(\theta), K_2$ and $x_1(\theta)$ in the Appendix A.2. Moreover, in optimum, changes in the co-state $\mu(\theta)$ with respect to skill types are equal to the negative effect of the state variable $W(\theta)$ upon the Hamiltonian $\mathcal{H}$; that is,

$$\dot{\mu}(\theta) = -\frac{\partial \mathcal{H}}{\partial W(\theta)} = -\pi(\theta)\left[1 - \frac{\lambda_2}{\beta u'(c_2(\theta))}\right]. \tag{5b}$$

Note that if the IC constraint is not binding and thus $\mu(\theta) = 0$, (5b) does not apply and the first-order conditions reduce to standard conditions in the Ramsey model. However, if the IC constraint binds and thus $\mu(\theta) \neq 0$, (5b) applies, and the first-order conditions differ from those in the Ramsey model.

Based on (5a) and (5b), in Appendix A.3 we have shown the following lemma.

Lemma 1. *Suppose that* $c_2(\theta)$ *is monotone increasing in* θ. *Then,* $\lambda_2 > 0$ *and* $\mu(\theta) < 0$ *for* $\theta \in (\underline{\theta}, \overline{\theta})$.

First, it is standard to obtain a positive multiplier λ_2 of the resource constraint. Moreover, according to (5a), the co-state of information frictions μ is zero at the top and the bottom of the type distribution, which is standard. However, outside the top and bottom types in the distribution, the co-state of the IC constraint is $\mu(\theta) < 0$, which suggests a welfare cost due to information frictions.[25]

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[24] The solution to the relaxed planning problem might not be a solution to the full program. Hence, for the proposed calibrations in Section 6 below, we have followed Farhi and Werning (2013) and Stantcheva (2017) and numerically verified that the allocation solved by our relaxed problem is incentive compatible and gives the utility intended by the planner.

[25] A negative co-state $\mu(\theta)$ may be viewed as the marginal welfare loss in order for the social planner to choose allocations that are incentive-compatible. If the IC constraint is not binding, which arises when agents have no incentives to cheat, then $\mu(\theta) = 0$ and the social planner does not have to sacrifice the welfare.

With $\mu(\theta) < 0$, we can rearrange the first-order conditions to obtain the following four conditions[26], which characterize the constrained efficient allocation in the social planner's problem.

$$\frac{u'\left(c_1(\theta)\right)}{\beta u'\left(c_2(\theta)\right)} - \frac{\mu(\theta)}{\lambda_2}\frac{u''\left(c_1(\theta)\right)\frac{\partial y_1(\theta)}{\partial\theta}}{\pi(\theta)} = F_k\left(K_2, Z_2\right) + 1 - \delta_k, \tag{6a}$$

$$\frac{\phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{h_1}}{u'\left(c_1(\theta)\right)}\left(1 - \frac{\mu(\theta)\phi'\left(\frac{z_1(\theta)}{h_1}\right)u''\left(c_1(\theta)\right)\frac{\partial y_1(\theta)}{\partial\theta}}{\lambda_1\pi(\theta)u'\left(c_1(\theta)\right)F_z\left(K_1, Z_1\right)h_1}\right)^{-1} = F_z\left(K_1, Z_1\right), \tag{6b}$$

$$\frac{\phi'\left(\frac{z_2(\theta)}{h_2}\right)\frac{1}{h_2}}{u'\left(c_2(\theta)\right)} - \frac{\beta\mu(\theta)}{\lambda_2\pi(\theta)\left[h_2(\theta)\right]^2}\left[\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + \phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\frac{\partial h_2(\theta)}{\partial\theta} = F_z\left(K_2, Z_2\right). \tag{6c}$$

$$u'\left(c_1(\theta)\right) = \frac{\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)u'\left(c_1(\theta)\right)\psi_x\left(x_1(\theta), y_1(\theta)\right)}{\left[F_k\left(K_2, Z_2\right) + 1 - \delta_k\right]u'\left(c_2(\theta)\right)\left[h_2(\theta)\right]^2} - \frac{\beta\mu(\theta)u'\left(c_1(\theta)\right)z_2(\theta)\psi_x\left(x_1(\theta), y_1(\theta)\right)}{\lambda_1\pi(\theta)\left[h_2(\theta)\right]^3}\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + 2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}. \tag{6d}$$

Condition (6a) connects the household's *marginal rate of substitution* (henceforth, MRS) between consumption today and tomorrow with the *marginal rate of transformation* (henceforth, MRT) between consumption and investment today. As investment today accumulates physical capital tomorrow, the MRT is the firm's *marginal product of capital* (henceforth, MPK) tomorrow. Moreover, (6b) and (6c) also link the household's MRS between leisure and consumption to the MRT between labor and consumption. This MRT is the firm's *marginal product of labor* (henceforth, MPL). Finally, (6d) links the marginal cost with the marginal benefit of observable education expenditure in human capital investment. These conditions determine the optimal implicit marginal tax rates on capital income and labor income, as well as the optimal implicit subsidy rate on observable human capital investment, which are also known as wedges that are defined in the next subsection.

3.2 Properties of the optimum and wedges

Wedges measure distortions in the second-best allocation relative to the laissez-faire allocation. Agents' work effort, consumption and non-verifiable education expenses are private information, which generate distortions. There are three marginal distortions in the second-best allocation, defined as the labor wedge τ_{z_t}, the capital wedge τ_{k_t} and the human capital wedge τ_{x_t} as follows.

$$\left(1 - \tau_{z_t}\right)u'\left(c_t\right) \equiv \phi'\left(\frac{z_t}{h_t}\right)\frac{1}{w_t h_t}, \tag{7a}$$

$$u'\left(c_{t-1}\right) \equiv \left(1 - \tau_{k_t}\right)\beta R_t E_{t-1}\left[u'\left(c_t\right)\right], \tag{7b}$$

$$\left(1 - \tau_{x_t}\right)u'\left(c_t\right) \equiv \beta\psi_x\left(x_t, y_t\right)E_{t-1}\left[\phi'\left(\frac{z_{t+1}}{h_{t+1}}\right)\frac{z_{t+1}}{\left(h_{t+1}\right)^2}\right], \tag{7c}$$

[26] The detail and the rearrangement of the first-order conditions are relegated in Appendix A.2.

where $w_t = F_Z(K_t, Z_t)$ and $R_t = F_k(K_t, Z_t) + (1 - \delta_k)$.

In the dynamic taxation literature, the labor wedge is an *intratemporal* wedge, which measures the difference of household's MRS between labor and consumption today from firm's MPL today (i.e., the wage rate). In a similar fashion, the capital wedge is an *intertemporal* wedge, which measures the difference of household's MRS between consumption today and tomorrow from firm's MPK tomorrow (i.e., the rental rate). As for the human capital wedge, it measures the gap between the marginal cost and the marginal benefit from observable education investment in human capital. The labor and capital wedges are defined as implicit labor and capital tax rates, which is standard in the dynamic taxation literature. Following Stantcheva (2017), the human capital wedge is defined as an implicit subsidy to observable human capital investment x_t. The labor, capital and human capital wedges serve to measure the distortions of the second-best allocation relative to the laissez-faire allocation. In the laissez-faire, these wedges would be zero. However, as consumption is indistinguishable from education expenditure, the information asymmetry distorts household's MRS between labor and consumption today and MRS between consumption today and tomorrow, which cause labor, (physical) capital and human capital wedges to deviate from zero. The signs of those wedges for agents with heterogenous types are studied in Propositions 2 to 4 below. The proofs of Propositions 2 to 4 and Corollaries 1 and 2 are relegated in Appendix A.4.

First, to derive the modified inverse Euler equation, we denote the following notation $\Omega_t(c_t, c_{t+1})$

$$\Omega_t(c_t, c_{t+1}) \equiv \frac{1}{u'(c_t)} - \frac{1}{\beta R_{t+1} u'(c_{t+1})}. \tag{7d}$$

Proposition 2 below determines the sign of the capital wedge, followed by a corollary for the capital wedge in the special case when there are only observable education expenses.

Proposition 2. *In the case of a separable utility, the modified inverse Euler equation is of the following form*

$$\frac{1}{u'(c_1(\theta))} = \frac{1}{\beta R_2\, u'(c_2(\theta))} + \frac{-\mu(\theta)u''(c_1(\theta))}{\lambda_1 \pi(\theta) u'(c_1(\theta))} \frac{\partial y_1(\theta)}{\partial \theta} \;\; for \;\; \theta \in (\underline{\theta}, \overline{\theta}), \tag{8a}$$

where $-\frac{\mu(\theta)u''(c_1(\theta))}{\lambda_1 \pi(\theta) u'(c_1(\theta))} \frac{\partial y_1(\theta)}{\partial \theta} > 0$ *is the HCI effect. Then, the capital wedge is positive and satisfies*

$$\tau_{k_2}(\theta) = u'(c_1(\theta)) \Omega_1(c_1(\theta), c_2(\theta)) > 0 \;\; for \;\; \theta \in (\underline{\theta}, \overline{\theta}).$$

Corollary 1. *If there are only verifiable education expenses, the (inverse) Euler equation holds; that is,*

$$\frac{1}{u'(c_1(\theta))} = \frac{1}{\beta R_2\, u'(c_2(\theta))} \;\; for \;\; \theta \in [\underline{\theta}, \overline{\theta}]. \tag{8b}$$

Then, capital wedge is zero: $\tau_{k_2}(\theta) = 0$ *for* $\theta \in [\underline{\theta}, \overline{\theta}]$

As there are non-verifiable education expenses, the modified inverse Euler equation (8a) holds, and the capital wedge is positive; i.e, $\tau_{k_2}(\theta) > 0$ for $\theta \in (\underline{\theta}, \overline{\theta})$. A positive capital tax comes from indistinguishable consumption and education expenditure, dubbed the *HCI effect*. The reason goes as follows. Even without time-varying skill shocks, due to indistinguishable consumption and education expenditure, agents may underreport their types by substituting away from education expenses toward consumption today, which distorts the MRS between consumption today and tomorrow. This gives rise to an indirect tax on future consumption, and thus a positive capital wedge. Specifically, by reducing education expenses for consumption today, agents have a higher consumption level today, which gives a stronger preference to save for smoothing consumption. However, as savings are observable, the deviators, who under-report skills, have to save as much as their reporting types to avoid being caught. A positive capital wedge hurts the deviators more than the truth-tellers, and thus offsets the benefit from under-reporting. So, it is an efficient way to provide the correct incentives for agents to reveal their true type. The same result holds in the extension to T periods later.

By contrast, if there are only verifiable education expenses, the standard consumption Euler equation (8b) holds and thus, the capital wedge is zero. This is an application of the Atkinson and Stiglitz (1976) result on the non-optimality of indirect taxes if the preference is separable in consumption and labor. Yet, the Atkinson and Stiglitz (1976) result cannot apply when there are non-verifiable education expenses.

Next, we have proved in the Appendix A.4 the following proposition and corollary concerning the sign of the labor wedge.

Proposition 3. *In the case of a separable utility, the labor wedge is negative in the first period, and becomes positive in the terminal period. To be more specific,*

$$\tau_{z_1}(\theta) = \underbrace{\frac{\mu(\theta)\phi'\left(\frac{z_1(\theta)}{h_1}\right)u''\big(c_1(\theta)\big)}{\lambda_1\pi(\theta)u'\big(c_1(\theta)\big)w_1h_1}\frac{\partial y_1(\theta)}{\partial\theta}}_{\textit{skill-fostering effect}} < 0 \;\; for\; \theta \in \left(\underline{\theta}, \overline{\theta}\right),$$

and

$$\tau_{z_2}(\theta) = \underbrace{\frac{-\beta\mu(\theta)}{\lambda_2\pi(\theta)[h_2(\theta)]^2\mathrm{w}_2}\left[\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + \phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\frac{\partial h_2(\theta)}{\partial\theta}}_{\textit{shirk-preventing effect}} > 0 \; for\; \theta \in \left(\underline{\theta}, \overline{\theta}\right).$$

Corollary 2. *If there are only verifiable education expenses,* $\tau_{z_1}(\theta) = 0$ *and* $\tau_{z_2}(\theta) > 0$ *for* $\theta \in (\underline{\theta}, \overline{\theta})$.

Intuitively, when there are non-verifiable education expenses, consumption is indistinguishable from education expenditure. Then, high-skilled agents may underreport their skill types and reduce education expenses for consumption. A negative labor wedge in the first period acts like an implicit subsidy which is a mechanism to induce agents to work according to their true types and invest sufficiently on education, dubbed *skill-fostering* effect.

Yet, if there are only verifiable education expenses, agents cannot cut down education expenses without being caught. Then, given the same initial human capital, there is no distortion between consumption and labor in the first period and thus, the labor wedge is zero in the first period.

In the terminal period, an agent has no incentives to invest in human capital. This goes back the standard Mirrlees literature, the *shirking-preventing* effect. Thus, the labor tax is positive in order to prevent agents from shirking. In the extension to T periods, we will show a transition from the *skill-fostering* effect to the *shirking-preventing* effect in early periods, and thus labor wedge is negative early in the life cycle.

Finally, for the human capital wedge, the definition in (7c) may not reflect the distortion purely caused by observable education expenses. As Stantcheva (2017) pointed out, the human capital wedge may include several simultaneous distortions, such as labor distortions and capital distortions. To measure the distortion purely caused by observable education expenses, Stantcheva (2017) undid the part of the effects of labor and capital distortions to find a measure of the net distortion on observable human capital expenses. Following Stantcheva (2017), the net human capital wedge is defined as follows.

Definition 2. The net wedge on observable human capital expenses, $\tau_{x_t}^n$ is defined as

$$\tau_{x_t}^n \equiv E_t\left[\tau_{x_t} - \mathcal{N}_{t+1} + \mathcal{K}_{t+1}\right],$$

where

$$\mathcal{N}_{t+1} = \frac{w_{t+1} z_{t+1}}{R_{t+1} h_{t+1}} \psi_x(x_t, y_t) \tau_{z_{t+1}}$$

captures the distortion caused by the (appropriately scaled) labor wedge $\tau_{z_{t+1}}$, and

$$\mathcal{K}_{t+1} = \frac{\beta z_{t+1}}{(h_{t+1})^2} \phi'\left(\frac{z_{t+1}}{h_{t+1}}\right) \psi_x(x_t, y_t) \Omega_t$$

captures the distortion caused by the capital wedge, where Ω_t is related to the modified inverse Euler equation $\Omega_t = \frac{1}{u'(c_t)} - \frac{1}{\beta R_{t+1} u'(c_{t+1})}$ defined in (7d).

In Appendix A.4, we have proved the following proposition regarding the net human capital wedge.

Proposition 4. *In the case of a separable utility, the net human capital wedge $\tau_{x_1}^n(\theta)$ is given as follows:*

$$\tau_{x_1}^n(\theta) = \frac{-\beta \mu(\theta) z_2(\theta)}{\lambda_1 \pi(\theta) (h_2(\theta))^3} \phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right) \psi_x\big(x_1(\theta), y_1(\theta)\big) \frac{\partial h_2(\theta)}{\partial \theta} > 0 \quad for\ \theta \in (\underline{\theta}, \overline{\theta}).$$

The result of a positive net human capital wedge is not new. Stantcheva (2017) established that the net human capital wedge is positive if and only if the Hicksian coefficient of complementarity between ability and human capital is less than one. As the human capital technology is set to be separable (cf. Assumption 1) in our paper, the Hicksian coefficient is zero.[27] Hence, our positive human capital wedge is consistent with the result in Stantcheva (2017).

Based on the boundary condition (5a), it is clear that the capital wedge, the labor wedge and the human capital wedge are all zero for agents at the top and the bottom of the type distribution: $\tau_{k_2}(\overline{\theta}) =$

[27] The positive net human capital wedge is enhanced if we allow for the complementarity between human capital investment and skill types.

$\tau_{k_2}(\underline{\theta}) = 0,\ \tau_{z_t}(\overline{\theta}) = \tau_{z_t}(\underline{\theta}) = 0$ for $t = 1,2,$ and $\tau_{x_1}(\overline{\theta}) = \tau_{x_1}(\underline{\theta}) = 0.$ This confirms that the result of "no distortion at the top and the bottom" obtained in static models (e.g., Mirrlees, 1971; Stiglitz, 1982) and dynamic models with exogenous skills (e.g., Farhi and Werning, 2013), is robust in dynamic models with unobservable human capital investment over time.[28]

4. A T-period Model

We now generalize our results from two periods to T periods. To make our T-period problem tractable, following Stantcheva (2017), we focus on partial equilibrium, wherein the interest rate R_t and the wage rate w_t are treated as predetermined.[29] A T-period model is more complicated than a 2-period model, as skill shocks that an agent has experienced earlier affect human capital investment later. Denote by $\theta^t \equiv (\theta_1, \theta_2, \ldots, \theta_t)$ the history of skill shocks up to period t. Then, an agent's choice in period t is affected not only by skill shocks θ_t realized in period t but also by the history of skill shocks θ^{t-1} undergone before period t. Then, the evolution of the human capital is:

$$h_{t+1}(\theta^t) = \psi\left(x_t(\theta^t), y_t(\theta^t)\right) + \theta_t. \tag{9}$$

To simplify the model, we assume that human capital is completely depreciated in one period.[30] If the human capital is not depreciated after one period, the educational investment in the current period will affect not only human capital in the next period but also human capital after the next period until the terminal period. Then, it is too complicated to analyze the T-period model. Although the depreciation of human capital after one period is somewhat restrictive, the assumption greatly reduces the complexity of the determination of the education choice (cf. Proposition 5 below), so as to focus on the interaction between the current period's educational investment and the next period's human capital level. Then, the planner's problem can be stated in a tractable recursive formulation by focusing on the state variables in the current period, rather than the whole history of types, since, for any period t, once we condition on the history of shocks in a period earlier θ_{t-1}, the entire history of shocks θ^{t-2} is redundant.

The lifetime utility of an agent with type history θ^t is written by the following Bellman equation:

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[28] Once we allow the income distribution to have a Pareto tail (Saez, 2001), or if there is moving support with the upper and lower bounds depending on the past type realization (Farhi and Werning, 2013), the "no distortion at the top" result no longer holds.

[29] Stantcheva (2017) focused on partial equilibrium, who specified a fixed interest rate with a pre-determined wage rate being a function of human capital in her T-period model. In our two-period model earlier, the interest rate and the wage rate are derived from aggregate production function, which depends on physical and effective labor. To be consistent with our two-period model, we specify an interest rate and a wage rate in our T-period model, such that physical capital yields a pre-determined interest rate and the product of the wage rate and human capital is endogenous and equal to the marginal return to labor.

[30] Although we cannot analytically solve the model if the human capital is not depreciated after one period, the qualitative results are the same in that the capital wedge is determined by the HCI effects in the current and the next periods, and the labor wedge by the shirk-preventing effect and the skill-fostering effect, and moreover, the net human capital wedge is still positive.

$$W(\theta^t) = u(c(\theta^t)) - \phi\left(\frac{z(\theta^t)}{h(\theta^{t-1})}\right) + \beta\int W(\theta^{t+1})\pi(\theta_{t+1})d\theta_{t+1}.$$

In a T-period model, agents report their types in each period. We denote the reporting strategy by $\sigma \equiv (\sigma_1(\theta^T),\ldots,\sigma_T(\theta^T))$ when an agent with type $\theta^T = (\theta_1,\ldots,\theta_T)$ specifies a reported type σ_t for period t. Denote the set of all possible reporting strategies by $\mathcal{R}$. For observable allocations like effective labor, verifiable education expenses, and the sum of consumption and non-verifiable education expenses, the planner can directly assign the allocations as functions of the history of reported types $\sigma^t \equiv (\sigma_1,\ldots,\sigma_t)$. But, for unobservable allocations like non-verifiable education expenses and consumption, an agent will choose the optimal allocation based on their reporting strategy $\sigma \in \mathcal{R}$. To analyze the constrained efficient allocations, we start by analyzing the choice of non-verifiable education expenses and consumption for the agent with a reporting strategy $\sigma \in \mathcal{R}$.

4.1 The Agent's Problem

In Appendix A.5, we have solved the agent's problem concerning the choice of non-verifiable education expenses and consumption, which is characterized as follows.

Proposition 5. *For a reporting strategy $\sigma \in \mathcal{R}$, the optimal non-verifiable human capital investment is determined by the following condition*

$$u'(c^\sigma(\theta^t)) = -\beta\int \phi_h\left(\frac{z(\sigma^{t+1})}{h^\sigma(\theta^t)}\right)\psi_y\left(x(\sigma^t), y^\sigma(\theta^t)\right)\pi(\theta_{t+1})d\theta_{t+1}.$$

Thus, the agent's optimal choice characterized in Proposition 1 is extended to T periods.

For a type θ^t agent, the expected lifetime utility in period t under a truth-telling strategy is

$$W(\theta^t) = u(c(\theta^t)) - \phi\left(\frac{z(\theta^t)}{\psi(x(\theta^t), y(\theta^t)) + \theta_{t-1}}\right) + \beta\int W(\theta^{t+1})\pi(\theta_{t+1})d\theta_{t+1}, \tag{10a}$$

while the expected lifetime utility in period t under a reporting strategy $\sigma \in \mathcal{R}$ is

$$W^\sigma(\theta^t) = \max_{y^\sigma(\theta^t)} u(c^\sigma(\theta^t)) - \phi\left(\frac{z(\sigma^t)}{\psi(x(\sigma^{t-1}), y^\sigma(\theta^{t-1})) + \theta_{t-1}}\right) + \beta\int W^\sigma(\theta^{t+1})\pi(\theta_{t+1})d\theta_{t+1}. \tag{10b}$$

where $c^\sigma(\theta^t)$ must satisfy Proposition 5, when there are non-verifiable education expenses.

4.2 Incentive compatibility constraint

To study the incentive compatibility of the relaxed social planning problem, we follow Stantcheva (2017) and consider one particular deviation strategy $\hat{\sigma}$ with the reported type $\hat{\sigma} \equiv (\theta_1,\ldots,\hat{\theta}_t,..,\theta_T)$, wherein the agent reports strategies truthfully in all periods except t (i.e., $\hat{\sigma}_u(\theta^T) = \theta_u \ \forall u \neq t$) when he may deviate by specifying a reported type $\hat{\sigma}_t(\theta^T) = \hat{\theta}_t$. Denote the set of this particular deviation strategy $\hat{\sigma}$ by $\hat{R}_t(\theta^T) \equiv \{\hat{\sigma} = (\hat{\theta}_1, \ldots, \hat{\theta}_T) | \hat{\theta}_u = \theta_u \ \forall u \neq t, \ and \ \hat{\theta}_t \in \Theta\}$.[31]

In Appendix A.7, we have used Proposition 5 to prove the following properties under this kind of reporting strategy.

Proposition 6. *Consider one particular deviation strategy $\hat{\sigma} \in \hat{R}_t(\theta^T)$. Then, the optimal allocations have the following properties:*

(1) In deviation period t, the optimal non-verifiable human capital investment is decreasing with the true type θ_t, but the human capital level is still increasing with the true type θ_t. That is

$$\frac{\partial y^{\hat{\sigma}}(\theta^t)}{\partial \theta_t} < 0 \ and \ \frac{\partial h^{\hat{\sigma}}(\theta^t)}{\partial \theta_t} > 0.$$

(2) For a period s after the deviation period (i.e., $s \geq t+1$), given the same reporting type $\theta^T = (\theta_1, \ldots, \theta_T)$, the deviating agent's optimal non-verifiable human capital investment and consumption are the same as the truth-telling agent. That is,

$$y^{\hat{\sigma}}(\theta^s) = y(\hat{\theta}^s) \ \ and \ \ c^{\hat{\sigma}}(\theta^s) = c(\hat{\theta}^s).$$

Property (1) in Proposition 6 is an extension of Proposition 1 with 2 periods. Based on Proposition 6, for any period $s \geq t+2$, it is straightforward to show that a type θ^s agent with a reported type $\hat{\theta}^s$ obtains the same expected lifetime utility as that of a truth-telling agent, $W^{\hat{\sigma}}(\theta^s) = W(\hat{\theta}^s)$; that is,

$$\begin{aligned} W^{\hat{\sigma}}(\theta^s) &= u\left(c^{\hat{\sigma}}(\theta^s)\right) - \phi\left(\frac{z(\hat{\theta}^s)}{\psi(x(\hat{\theta}^{s-1}), y^{\hat{\sigma}}(\hat{\theta}^{s-1})) + \theta_{s-1}}\right) + \beta \int W^{\hat{\sigma}}(\theta^{s+1}) \pi(\theta_{s+1}) d\theta_{s+1} \\ &= u\left(c(\hat{\theta}^s)\right) - \phi\left(\frac{z(\hat{\theta}^s)}{\psi(x(\hat{\theta}^{s-1}), y(\hat{\theta}^{s-1})) + \hat{\theta}_{s-1}}\right) + \beta \int W(\hat{\theta}^{s+1}) \pi(\hat{\theta}_{s+1}) d\hat{\theta}_{s+1} = W(\hat{\theta}^s), \ \forall \ s \geq t+2. \end{aligned}$$

Given $W^{\hat{\sigma}}(\theta^s) = W(\hat{\theta}^s)$ for any $s \geq t+2$ and $c^{\hat{\sigma}}(\theta^{t+1}) = c(\hat{\theta}^{t+1})$, the expected lifetime utility in period t of the agent with a reporting strategy $\hat{\theta}^s$ can be rewritten as follows.

$$\begin{aligned} W^{\hat{\sigma}}\left(\theta^t\right) &= u\left(c\left(\hat{\theta}^t\right) + y\left(\hat{\theta}^t\right) - y^{\hat{\sigma}}\left(\theta^t\right)\right) - \phi\left(\frac{z(\hat{\theta}^t)}{\psi(x(\hat{\theta}^{t-1}), y(\hat{\theta}^{t-1})) + \theta_{t-1}}\right) \\ &+ \beta \int \left[u\left(c\left(\hat{\theta}^{t+1}\right)\right) - \phi\left(\frac{z(\hat{\theta}^{t+1})}{\psi(x(\hat{\theta}^t), y^{\hat{\sigma}}(\theta^t)) + \theta_t}\right) + \beta \int W\left(\hat{\theta}^{t+2}\right) \pi\left(\theta_{t+2}\right) d\theta_{t+2}\right] \pi\left(\theta_{t+1}\right) d\theta_{t+1}. \end{aligned} \tag{11a}$$

Like (4c) in the two-period model, the incentive compatibility constraint is

$$W\left(\theta^t\right) = \max_{\hat{\theta}_t \in \Theta} W^{\hat{\sigma}}\left(\theta^t\right). \tag{11b}$$

Denoting $\psi_y(\theta) = \psi_y(x(\theta), y(\theta))$, then differentiating (11a) with respect to θ_t gives an envelope condition that includes a static utility gain and a dynamic utility gain, like (4d) in the two-period model.

[31] In the text, we only focus on a one-deviation strategy, which is the standard way of the *first-order approach*. In Appendix A.6, we show that, under a proper condition, the incentive compatibility of the one-deviation strategy can be extended to a multi-deviation strategy.

$$\dot{W}\left(\theta^{t}\right)=-u'\left(c\left(\theta^{t}\right)\right)\frac{\partial y\left(\theta^{t}\right)}{\partial\theta_{t}}+\beta\int\phi'\left(\frac{z\left(\theta^{t+1}\right)}{h_{t+1}\left(\theta^{t}\right)}\right)\frac{z\left(\theta^{t+1}\right)}{\left[h_{t+1}\left(\theta^{t}\right)\right]^{2}}\frac{\partial h\left(\theta^{t}\right)}{\partial\theta_{t}}\pi\left(\theta_{t+1}\right)d\theta_{t+1}. \tag{11c}$$

The expected lifetime utility of truth-telling agents with type θ^t and the envelope condition can be expressed in terms of the following recursive formulation:

$$W\left(\theta^{t}\right)=u\left(c\left(\theta^{t}\right)\right)-\phi\left(\frac{z\left(\theta^{t}\right)}{h\left(\theta^{t-1}\right)}\right)+\beta v\left(\theta^{t}\right), \tag{11d}$$

$$\dot{W}\left(\theta^{t}\right)=-u'\left(c\left(\theta^{t}\right)\right)\frac{\partial y\left(\theta^{t}\right)}{\partial\theta_{t}}+\beta\Delta\left(\theta^{t}\right),$$

where

$$v\left(\theta^{t}\right)=\int W\left(\theta^{t+1}\right)\pi\left(\theta_{t+1}\right)d\theta_{t+1},$$

$$\Delta\left(\theta^{t}\right)=\beta\int\phi'\left(\frac{z\left(\theta^{t+1}\right)}{h\left(\theta^{t}\right)}\right)\frac{z\left(\theta^{t+1}\right)}{\left[h\left(\theta^{t}\right)\right]^{2}}\frac{\partial h\left(\theta^{t}\right)}{\partial\theta_{t}}\pi\left(\theta_{t+1}\right)d\theta_{t+1},$$

and these two new variables $v(\theta^t)$ and $\Delta(\theta^t)$ will serve as state variables.

4.3 The relaxed planning problem and properties of the optimum and wedges

Now, we analyze a family of related problems that admit a recursive dynamic programing problem through a suitable definition of state variables. In a T-period model, the utility maximization problem is subject to period-by-period resource constraints. To avoid imposing so many resource constraints, we follow Atkeson and Lucas (1992), who studied a dual continuation problem, wherein the partial equilibrium of the dynamic incentive problems was analyzed without period-by-period resource constraints imposed upon the principal.[32] Given period t and past history θ^{t-1}, we envisage the dual continuation problem that minimizes the remaining expected discounted resource costs, while taking as given previous values for state variables $v(\theta^{t-1})$ and $\Delta(\theta^{t-1})$.

The remaining expected discounted resource cost of providing the allocation is:

$$\mathcal{K}\left(v,\Delta,h,\theta_{s-1},s\right)=\min\sum_{t=s}^{T}\frac{1}{R_{s+1}R_{s+2}\cdots R_{t}}\int_{\Theta^{t}}\left[c\left(\theta^{t}\right)+x\left(\theta^{t}\right)+y\left(\theta^{t}\right)-w_{t}z\left(\theta^{t}\right)\right]\pi\left(\theta_{t}\right)\pi\left(\theta_{t-1}\right)\ldots\pi\left(\theta_{s}\right)d\theta_{t}\ldots d\theta_{s}.$$

The social planner minimizes the expected discounted cost above, subject to the incentive compatibility condition and the expected lifetime utility for each (initial) type θ being above a targeted threshold value. For any period t, once we condition on the history of shocks in one period earlier θ_{t-1}, the entire history of shocks θ^{t-2} is redundant. Then, using the recursive formulation in (11d), Appendix A.8 has written down the *T*-period relaxed social planning problem for periods $t = 1, 2, 3, \ldots, T-1$, in terms of a recursive Bellman equation. The details of solving the relaxed social planning problems relegated in Appendix A.8.

We now analyze the properties of the optimum in terms of the sign of the capital wedge and the

[32] The same approach was used by Farhi et al (2012), Farhi and Werning (2013) and Stantcheva (2017).

labor wedge in the T-period model. As the wedges depend on the history of types, the definitions in (7a)-(7c) are revised as follows.

$$\left(1-\tau_z\left(\theta^t\right)\right)u'\left(c\left(\theta^t\right)\right)\equiv\phi'\left(\frac{z\left(\theta^t\right)}{h\left(\theta^{t-1}\right)}\right)\frac{1}{w_t h\left(\theta^{t-1}\right)}, \tag{12a}$$

$$u'\left(c\left(\theta^t\right)\right)\equiv\left(1-\tau_k\left(\theta^t\right)\right)\beta R_{t+1}E_t\left[u'\left(c\left(\theta^{t+1}\right)\right)\right], \tag{12b}$$

$$\left(1-\tau_x\left(\theta^t\right)\right)u'\left(c\left(\theta^t\right)\right)\equiv\beta\psi_x\left(x\left(\theta^t\right),y\left(\theta^t\right)\right)E_t\left[\phi'\left(\frac{z\left(\theta^{t+1}\right)}{h\left(\theta^t\right)}\right)\frac{z\left(\theta^{t+1}\right)}{\left[h\left(\theta^t\right)\right]^2}\right]. \tag{12c}$$

Let $\hat{\mu}(\theta)$ be the co-state variable associated with the envelope condition $\dot{W}(\theta)$ and $\gamma(\theta_-)$ be the shadow price associated of the state variable Δ in the relaxed planning problem. In Appendix A.8, we have shown that $\hat{\mu}(\theta)>0$ and $\gamma(\theta_-)<0$.[33]

First, in Proposition 7 and Corollary 3 below, we establish the modified inverse Euler equation and the sign of the capital wedge. The proof is relegated in Appendix A.8.

Proposition 7. *In the case of separable utility, the modified inverse Euler equation is of the form*

$$\frac{1}{u'\left(c(\theta^{t-1})\right)}=E_{t-1}\left[\frac{1}{\beta R_t u'\left(c(\theta^t)\right)}+\Omega_{t-1}(c(\theta^{t-1}),c(\theta^t))\right], for\ \ \theta^t\in\Theta^t,\ \ t=2,3,..T, \tag{13a}$$

where

$$E_{t-1}[\Omega_{t-1}(c(\theta^{t-1}),c(\theta^t))]\equiv\underbrace{\frac{\hat{\mu}(\theta^{t-1})u''\left(c(\theta^{t-1})\right)}{\pi(\theta^{t-1})u'\left(c(\theta^{t-1})\right)}\frac{\partial y(\theta^{t-1})}{\partial\theta^{t-1}}}_{\text{Current period's HCI effect}}+\underbrace{\frac{-1}{\beta R_t}E_{t-1}\left[\frac{\hat{\mu}(\theta^t)u''\left(c(\theta^t)\right)}{\pi(\theta^t)u'(c(\theta^t))}\frac{\partial y(\theta^t)}{\partial\theta^t}\right]}_{\text{Next period's HCI effect}}. \tag{13b}$$

In particular, in the terminal period, as $\frac{\partial y(\theta^{T-1})}{\partial\theta^{T-1}}<0$, $\frac{\partial y(\theta^T)}{\partial\theta^T}=0$, *and thus,* $\Omega_{t-1}(c(\theta^{t-1}),c(\theta^t)>0$ *and*

$$\frac{1}{u'\left(c(\theta^{T-1})\right)}>\frac{1}{\beta R_T}E\left[\frac{1}{u'\left(c(\theta^T)\right)}\right],$$

which induces a larger capital wedge than that in the case when the standard inverse Euler equation holds.

When non-verifiable education expenses are not present, the modified inverse Euler equation (13a) reduces to the tradition inverse Euler equation. It is stated formally in Corollary 3 as follows.

Corollary 3. *If there are only verifiable education expenses,* $\frac{\partial y(\theta^{t-1})}{\partial\theta^{t-1}}=\frac{\partial y(\theta^t)}{\partial\theta^t}=0$ *and then the inverse Euler equation holds. That is,* $E_{t-1}[\Omega_{t-1}(c(\theta^{t-1}),c(\theta^t))]=0$.

[33] The sign of $\hat{\mu}(\theta)>0$ in Section 4 here is different from the sign of $\mu(\theta)<0$ in Section 3. The reason is that the planning problem in Section 4 is in terms of a social cost minimization, wherein the social cost is raised in order to be incentive compatible, while the planning problem in Section 3 is in terms of a social welfare maximization, wherein the social welfare is decreased in order to be incentive compatible.

Thus, the sign of the capital wedge characterized in Proposition 2 for two periods is extended to T periods. If there are only verifiable education expenses, Corollary 3 shows that the standard inverse Euler equation holds. Then, due to time-varying skill shocks, the capital wedge is positive ($\tau_{k_t}(\theta^t) > 0$), which is the standard outcome in the dynamic Mirrlees literature resulting from an insurance effect.

In contrast, when there are non-verifiable education expenses, due to inseparable consumption and education expenditure, $E_{t-1}[\Omega_{t-1}(c(\theta^{t-1}), c(\theta^t)] \neq 0$. Then, the standard inverse Euler equation does not hold. In this case, agents have incentives to reduce non-verifiable education expenses for consumption, thus a HCI effect. As is clear from our proof in the Appendix, a higher skill shock today exerts two HCI effects that offset each other. One effect is from reducing non-verifiable education expenses toward current period's (period $t-1$) consumption, which enhances the otherwise positive capital wedge from the insurance effect. By contrast, the other effect is via reducing non-verifiable education expenses toward next period's (period t) consumption, which offsets the otherwise positive capital wedge from the insurance effect. The net effect on the capital wedge is ambiguous, as it is not sure whether the next period's HCI effect is strong enough to dominate the sum of the current period's HCI effect and the insurance effect. Yet, in the terminal period, there is only the current period's HCI effect, so the capital wedge is unambiguously larger than the otherwise positive capital wedge arising from the insurance effect.

Next, we establish the sign of the labor wedge as follows with proof relegated in Appendix A.8.

Proposition 8. *In the case of a separable utility, the labor wedge is negative in the first period,* $\tau_z(\theta) < 0$ *for* $\theta \in (\underline{\theta}, \overline{\theta})$*, and* $\tau_z(\theta^t)$ *is ambiguous for* $t = 2,3,..T-1$. *In the terminal period, the labor wedge is positive* $\tau_z(\theta^T) > 0$. *To be more specific,*

$$\tau_z(\theta^t) = \underbrace{\frac{-\gamma(\theta^{t-1})\frac{\partial h(\theta^{t-1})}{\partial \theta_{t-1}}\left[\phi'\left(\frac{z(\theta^t)}{h(\theta^{t-1})}\right) + \frac{z(\theta^t)}{h(\theta^{t-1})}\phi''\left(\frac{z(\theta^t)}{h(\theta^{t-1})}\right)\right]}{w_t[h(\theta^{t-1})]^2}}_{\textit{shirk-preventing effect} \begin{cases} = 0 & when\ t=1 \\ > 0 & otherwise \end{cases}} + \underbrace{\frac{-\hat{\mu}(\theta^t)u''\left(c(\theta^t)\right)\phi'\left(\frac{z(\theta^t)}{h(\theta^{t-1})}\right)}{\pi(\theta^t)u'\left(c(\theta^t)\right)w_t h(\theta^{t-1})}\frac{\partial y(\theta^t)}{\partial \theta^t}}_{\textit{skill-fostering effect} \begin{cases} = 0 & when\ t=T \\ < 0 & otherwise \end{cases}}. \tag{14}$$

Note that at $t = 1$, *only skill-fostering effect is present as agents are endowed with identical human capital in the first period, while at* $t = T$, *there is no need to invest in human capital, and therefore* $\frac{\partial y(\theta^T)}{\partial \theta^T} = 0$. *Hence, only shirk-preventing effect is present at terminal period.*

Corollary 4. *If there are only verifiable education expenses,* $\frac{\partial y(\theta)}{\partial \theta} = 0$. *Then, the labor wedge is zero in the first period,* $\tau_z(\theta) = 0$ *for* $\theta \in [\underline{\theta}, \overline{\theta}]$ *and positive in the rest of the periods,* $\tau_z(\theta^t) > 0$ *for* $t \geq 2$.

Thus, the sign of the labor wedge characterized in Proposition 3 for two periods is extended to T periods. If there are only verifiable education expenses, then $\frac{\partial y(\theta)}{\partial \theta} = 0$, and thus, as stated in Corollary 4, the labor wedge is zero in the first period and positive in all other periods. Hence, like the existing literature, there is only a shirk-preventing effect after period 1, and thus the labor wedge is positive in order to prevent agent from shirking. With non-verifiable education expenses, consumption expenses are indistinguishable from education and there is a skill-fostering effect. Then, in period 1, there is only a skill-fostering effect, and thus a negative labor wedge, or a subsidy to labor supply, is optimal. After period 1, the skill-fostering effect dominates the shirk-preventing effect in early periods. Hence, a negative labor wedge early in the life cycle is optimal, which is a mechanism to induce agents to work according to their true types and invest sufficiently on education. In later life cycle, human capital investment decreases and the skill-fostering effect phases out, so the labor wedge is positive.

To summarize these wedges, firstly, our positive capital wedge arises not only from an insurance effect due to time-varying skill shocks, but also from an HCI effect due to indistinguishable consumption from education expenses. This is a new mechanism, which is different from that in the existing Mirrlees models with exogenous skills (e.g., Diamond and Mirrlees, 1978; Golosov et al., 2003; Goloslov et al. 2006; Werning, 2007; Farhi and Werning, 2013), with observable human capital investment in Stantcheva (2017), and with unobservable human capital investment in Grochulski and Piskorski (2010). Moreover, the result adds value to Stantcheva (2014) in that we analytically separate today's positive HCI effect from tomorrow's negative HCI effect on the capital wedge, so as to assure when the capital wedge is larger or smaller than the capital wedge in the case with only observable human capital.

Secondly, our negative labor wedge early in the life cycle is a new result in the dynamic Mirrlees literature. This is different from the positive labor wedge in models with exogenous skills (e.g., Golosov et al., 2006; Werning, 2007; Farhi and Werning, 2013) and the model with observable human capital investment in Stantcheva (2017). Our result is also different from the model with unobservable human capital investment at the beginning of the life cycle by Grochulski and Piskorski (2010), wherein their labor wedge is positive for low-skill agents and negative for high-skill agents. The labor wedge in these existing studies serves to induce agents to work according to their types, thus a shirk-preventing effect. With indistinguishable consumption from education expenses, our negative labor wedge helps encourage agents to work more. Intuitively, the deviation strategies involve shirking and under-investing in human capital. The deviators are over-skilled relative to the truth-tellers, who provide the same low effective labor supply. As the effect from under-investing in human capital dominates the effect from shirking, the deviators have higher consumption relative to leisure, and thus have a stronger preference for leisure and a weaker preference for consumption than the truth-tellers. It is not worthwhile for the deviators to under-invest in human capital for more consumption. A marginal subsidy to labor income makes it optimal to provide the effective labor supply and invest in human capital according to their true types, thus a skill-fostering effect. The result adds value to Stantcheva (2014) as well. We obtain a negative labor wedge in early periods of the life cycle, at least unambiguously in the first period, when the negative skill-fostering effect on the labor wedge dominates the positive shirk-preventing effect.

Finally, as noted in section 2, the human capital wedge defined in (12c) may be affected by capital or labor distortions. Hence, it is necessary to define a net human capital wedge that reflects only the distortion caused by observable education expenses. The net human capital wedge defined in Definition 2 still applies in the general T-period model. In Appendix A.8, we prove the following proposition, which states that the net human capital wedge is positive, the same as the simple model in Section 2.

Proposition 9. *In the case of a separable utility, the net human capital wedge is positive as follows.*

$$\tau_x^n(\theta^t) = E_t\left[\frac{-\gamma(\theta^t)z(\theta^{t+1})}{R_{t+1}(h(\theta^t))^3}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\psi_x\left(x_t(\theta^t), y_t(\theta^t)\right)\frac{\partial h(\theta^t)}{\partial\theta_t}\right] > 0.$$

Remark. Propositions 7-9 have established the forms of the capital wedge, the labor wedge, and the net human capital wedge in the general T-period model with random types. By setting $T = 2$ and adjusting multipliers, such as setting $\hat{\mu} = \frac{-\mu}{\lambda_1}$ and $\gamma = \frac{\beta\mu}{\lambda_2\pi}$, these forms are reduced to those in Propositions 2-4 in the simple two-period model.[34] Multipliers in the general T-period model are different from those in the simple model and thus need adjustments, because the social planner problem is a cost minimization form in general model, while it is a welfare maximization form in the simple model.

5. Tax Implementation in an Equilibrium

While it is tempting to interpret the capital and labor wedges defined in (12a) and (12b) as capital and labor taxes, because there is a *double deviation* problem,[35] the relationship between wedges and taxes is not straightforward.[36] In this section, we build a tax system to implement the associated constrained efficient allocation in a decentralized economy. Such tax implementations usually are not unique. In general, there are different tax systems that can implement constrained efficient allocations as an equilibrium in a decentralized economy. See, e.g., Albanesi and Sleet (2006), Kocherlakota (2005) and Grochulski and Piskorski (2010). In our implementation, we show that the capital wedge and labor wedge are not just implicit marginal tax rates, but they are also explicit marginal tax rates in our tax system.

Golosov et al. (2006) pointed out that the simplest method of implementation is to assign arbitrarily high punishments if an agent's observable allocation in any period is different from the constrained efficient allocation. Yet, this way severely limits an agent's choices and may be unrealistic. To relax the limitation and to create a direct connection between wedges and taxes, we provide a simple tax system

[34] When $T = 2$, the history of type is $\theta^t = (\theta_1, \theta_2)$, but as there is no need to invest human capital in period 2 for period 3, θ_2 does not affect the economy and thus, is redundant. Hence, by setting $\theta_1 = \theta$ and adjusting multipliers, Propositions 7-9 reduce to Propositions 2-4, respectively.

[35] Intuitively, each wedge controls only one aspect of a worker's behavior (labor in a period, or savings) taking all other choices fixed *at the optimal level*. For example, assuming that an agent supplies the socially optimal amount of labor, a capital tax defined by a capital wedge would ensure that the agent also makes a socially optimal amount of savings. However, agents choose labor and savings jointly; if an agent considers changing her labor, then, in general, she also considers changing her savings. Because the wedges are not constant values, joint changes in savings and labor may change the wedges to other values, which possibly gives allocations that are better than socially optimal allocations. Thus, there are *double deviations*. Kocherlakota (2005) and Albanesi and Sleet (2006) showed that such double deviations would give an agent a higher utility than the utility from the socially optimal allocations, and therefore the optimal tax system must be enriched with additional elements in order to implement the optimal allocations.

[36] Moreover, the wedges may also differ from taxes because of the general-equilibrium effects, but these effects are shut down in the model.

with linear income taxes to implement the constrained efficient allocation in a market economy. Different from Grochulski and Piskorski (2010), deferred capital taxes are not necessary for our linear capital income taxes. We show that these optimal linear capital and labor income tax rates are exactly the same as the optimal wedges established by the social planner.

5.1 A class of the tax system

Our tax system is described as follows. The tax system $\{\mathcal{T}_t\}$ includes linear labor and capital income tax rates ($\hat{\tau}_z$ and $\hat{\tau}_k$) and lump-sum taxes (Γ). Note that the government observes verifiable education expenses, effective labor and capital $(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$, but cannot observe non-verifiable human capital investment. These linear labor and capital income tax rates thus depend on agents' history of verifiable education expenses, with the tax rates being $\hat{\tau}_z(\tilde{x}^t)$ and $\hat{\tau}_k(\tilde{x}^t)$ which are different for different agents. Moreover, the lump-sum tax depends on these three observable allocations,[37] since the lump-sum tax $\Gamma(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$ is different for different agents.

Therefore, for each period t, the tax revenue from an agent with observable allocation $(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$ in our tax system is

$$\mathcal{T}_t \equiv \Gamma\left(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1}\right) + \hat{\tau}_z\left(\tilde{x}^t\right) w_t \tilde{z}_t + \hat{\tau}_k\left(\tilde{x}^{t-1}\right) R_t \tilde{k}_t. \tag{15a}$$

5.2 Income condition and reduced forms of taxes

Since a part of human capital investment is verifiable, our tax system punishes agents whose verifiable human capital investment deviates from the constrained efficient allocation. Let the set of recommended verifiable education expenses be $X^t \equiv \{x^t : \exists \hat{\theta}^t \in \Theta^t \ s.t.\ x^t = x^t(\hat{\theta}^t)\}$, and agents are required to choose one of the allocations in X^t; if otherwise, they will face severe punishment. As for capital and effective labor, agents are not obligated to choose constrained efficient allocations, but are expected to make sure agents' after-tax income consistent with the constrained efficient allocation. Thus, we impose the following "income" condition:

$$S\left(x^t(\hat{\theta}^t), \tilde{z}_t, \tilde{k}_{t+1}\right) \equiv \left(1 - \hat{\tau}_z\left(\tilde{x}^t(\hat{\theta}^t)\right)\right) w_t \left(z(\hat{\theta}^t) - \tilde{z}_t\right) - \left(k(\hat{\theta}^t) - \tilde{k}_{t+1}\right) = 0, \text{ for all } \hat{\theta}^t \in \Theta^t. \tag{15b}$$

To avoid punishment, agents have to choose the observable allocation $(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$ such that $\tilde{x}^t = x^t(\hat{\theta}^t)$ and the income condition $S(x^t(\hat{\theta}^t), \tilde{z}_t, \tilde{k}_{t+1}) = 0$ is met for some $\hat{\theta}^t \in \Theta^t$; if otherwise, they face a severe punishment $\Gamma(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1}) = \infty$. Given such conditions, our tax system specifies the following reduced-form taxes for linear labor and capital income tax rates and lump-sum taxes.[38]

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[37] These three observable allocations $(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$ in the lump-sum tax Γ are used to check whether the income condition in Section 5.2 is satisfied or not. If the income condition is not met, then agents would be severely punished through the lump-sum tax. Once the income condition is met, then the lump-sum tax Γ is not affected by the value of these three observable allocations $(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$.

[38] Although Γ depends on the current labor z, it is a lump-sum tax, because once the income condition is satisfied, Γ depends on only the reporting type.

$$\begin{cases} \hat{\tau}_z\left(x^t(\hat{\theta}^t)\right) = \tilde{\tau}_z\left(\hat{\theta}^t\right) \\ \hat{\tau}_k\left(x^t(\hat{\theta}^t)\right) = \tilde{\tau}_k\left(\hat{\theta}^t\right) \\ \Gamma\left(x^t(\hat{\theta}^t), \tilde{z}_t, \tilde{k}_{t+1}\right) = \tilde{\Gamma}\left(\hat{\theta}^t\right) \qquad if \quad S\left(x^t(\hat{\theta}^t), \tilde{z}_t, \tilde{k}_{t+1}\right) = 0 \end{cases}$$

5.3 Implementation with the tax system

Given physical and human capital $\left(\tilde{k}_t, \tilde{h}_t\right)$ accumulated from previous periods, under our tax system $\{\mathcal{T}_t\}$, the problem of type θ_t agent in the decentralized economy in period t is:

$$\tilde{U}^t\left(\tilde{k}_t, \tilde{h}_t, \theta_t\right) = \max u\left(\tilde{c}_t\right) - \phi\left(\tfrac{\tilde{z}_t}{\tilde{h}_t}\right) + \beta E_t\left[\tilde{U}^{t+1}\left(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1}\right)\right],$$

subject to

$$\tilde{c}_t + \tilde{x}_t + \tilde{y}_t + \tilde{k}_{t+1} \le w_t \tilde{z}_t + R_t \tilde{k}_t - \mathcal{T}_t,$$

$$\tilde{h}_{t+1} = \psi\left(\tilde{x}_t, \tilde{y}_t\right) + \theta_t,$$

where $E_t\left[\tilde{U}^{t+1}\left(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1}\right)\right] = \int \tilde{U}^{t+1}\left(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1}\right)\pi(\theta_{t+1})d\theta_{t+1}$, and the maximization is taken over $\{\tilde{c}_t, \tilde{z}_t, \tilde{x}_t, \tilde{y}_t, \tilde{h}_{t+1}, \tilde{k}_{t+1}\}$. Note that the tax policies (15a) are substituted into the constraints above, and the income condition (15b) is used in solving the problem. In Appendix A.9, we have established the following proposition.

Proposition 10. *Under the tax system $\{\mathcal{T}_t\}$, the constrained efficient allocations can be implemented in a decentralized economy, and the corresponding two linear capital and labor income tax rates are consistent with the wedges. That is, $\tilde{\tau}_k(\theta^t) = \tau_k(\theta^t)$ and $\tilde{\tau}_z(\theta^t) = \tau_z(\theta^t)$, $t = 1, 2, \dots, T$.*

Proposition 10 says that the wedges (τ_z, τ_k) as defined in the planning problem in subsection 4.3 can implement the constrained efficient allocation in our tax system.

To help understand how the implementation works, it is useful to describe the implementation in the two-period model and, in particular, the key income condition (15b) in preventing the double deviation.

In our two-period model, to avoid severe punishment, agents have to choose the observable allocation $\left(\tilde{x}_1, \tilde{z}_1, \tilde{z}_2, \tilde{k}_2, \tilde{c}_2\right)$ such that $\tilde{x}_1 = x_1(\hat{\theta})$ and the income condition $S\left(x_1(\hat{\theta}), \tilde{z}_1, \tilde{k}_2\right) = 0$ is met for some $\hat{\theta} \in \Theta$, and (15b) reduces to a tax system of linear factor income tax rates as follows.

$$S_1^{\hat{\theta}}\left(\tilde{z}_1, \tilde{k}_2\right) \equiv \left(1 - \tau_{z_1}(\hat{\theta})\right) w_1\left(z_1(\hat{\theta}) - \tilde{z}_1\right) - \left(k_2(\hat{\theta}) - \tilde{k}_2\right) = 0,$$

$$S_2^{\hat{\theta}}\left(\tilde{c}_2, \tilde{k}_2\right) \equiv \left(1 - \tau_{k_2}(\hat{\theta})\right)\left(1 + r_2\right)\left(k_2(\hat{\theta}) - \tilde{k}_2\right) - \left(c_2(\hat{\theta}) - \tilde{c}_2\right) = 0,$$

where $1 + r_2 = R_2$.

The two-period tax system $\mathcal{T} = \{T_1, T_2\}$ is described as follows.[39]

[39] By restricting $x_1 = x_1(\theta)$ for some $\theta \epsilon \Theta$, without any further restrictions in Subsection 5.2 when the agent

In the first period, the taxes are

$$T_1 = \tilde{T}_1(\theta) \equiv \Gamma_1(\theta) + \tilde{\tau}_{z_1}(\theta) w_1 z_1,$$

if there is some $\theta \in \Theta$ such that the condition $S_1^{\theta}(z_1, k_2) = 0$ holds and $x_1 = x_1(\theta)$; otherwise, $T_1 = \infty$.

In the second period, the taxes are

$$T_2 = \tilde{T}_2(\theta) \equiv \Gamma_2(\theta) + \tilde{\tau}_{z_2}(\theta) w_2 z_2 + \tilde{\tau}_{k_2}(\theta) r_2 k_2,$$

if there is some $\theta \in \Theta$ such that the condition $S_1^{\theta}(z_1, k_2) = S_2^{\theta}(k_2, c_2) = 0$ holds; otherwise, $T_2 = \infty$.

The tax system is explained as follows. Linear labor tax rates and linear capital tax rates $(\tilde{\tau}_{z_t}(\theta), \tilde{\tau}_{k_2}(\theta))$ and lump-sum taxes $\Gamma_t(\theta)$, $t = 1, 2$ and $\theta \in (\underline{\theta}, \bar{\theta})$, are designed for agents who meet the two conditions $S_1^{\theta}(z_1, k_2) = S_2^{\theta}(k_2, c_2) = 0$. If any one of these two conditions is not met, then agents will be punished sufficiently severely.

Then, we establish the same results as those in Proposition 10 for two periods.[40] That is, there exists an optimal tax system $\mathcal{T} = \{T_1, T_2\}$ such that the linear factor income tax rates are consistent with the wedges. That is, $\tilde{\tau}_{k_2}(\theta) = \tau_{k_2}(\theta)$ and $\tilde{\tau}_{z_t}(\theta) = \tau_{z_t}(\theta)$ for $t = 1, 2$.

The result says that our tax system can implement the constrained efficient allocation as a competitive equilibrium. Moreover, these linear capital and labor tax rates in the competitive equilibrium $(\tilde{\tau}_{z_t}(\theta), \tilde{\tau}_{k_2}(\theta)), t = 1, 2,$ are consistent with the wedges $(\tau_{z_t}(\theta), \tau_{k_2}(\theta))$ in the planning problem.

5.4 The necessity of the income condition

In this subsection, we provide intuition concerning the role of the income condition (15b) that helps our tax system implement the constrained efficient allocation in our tax system.

Without the income condition (15b) in our tax system in a decentralized economy, the optimal allocation must satisfy the following Euler equation.

$$u'(\tilde{c}_t) = \beta R_{t+1}\left(1 - \tilde{\tau}_k\left(\hat{\theta}^t\right)\right) E\left[u'(\tilde{c}_{t+1})\right] \text{ for some } \hat{\theta}^t \in \Theta^t. \tag{16a}$$

In order to implement the constrained efficient allocation in this tax system, we must find an appropriate linear capital tax $\tilde{\tau}_k(\hat{\theta}^t)$ such that, for any reporting strategy $\hat{\sigma}(\theta^t) = \hat{\theta}^t$ in the planning problem, the allocations $\{c^{\hat{\sigma}}(\theta^t), c^{\hat{\sigma}}(\theta^{t+1})\}$ satisfy equation (16a).[41] However, this is impossible, because the government cannot tell shirking from truth-telling agents. For the same reporting type $\hat{\theta}^t$, shirking and truth-telling agents must have the same linear capital tax rate $\tilde{\tau}_k(\hat{\theta}^t)$. According to Proposition 6, under a particular deviated reporting strategy $\hat{\sigma} \in \hat{R}_t(\theta^T)$, both shirking and truth-telling

chooses allocations $\{c_1, c_2, y_1, h_2, z_1. z_2, k_2\}$ to maximize the lifetime utility, it is impossible that the resulting allocations are exactly the same as constrained efficient allocations. By adding the two extra constraints $S_1^{\theta}(z_1, k_2) = S_2^{\theta}(k_2, c_2) = 0$, the resulting allocations are the same as constrained efficient allocations. See Appendix A.10.

[40] The details of this two-period tax implementation can be found in Appendix A.10.

[41] Grochulski and Piskorski (2010) pointed out that if (16a) does not hold for some reporting strategies, then this reporting strategy is not individually optimal and complicated an equilibrium strategy, and thus the constrained efficient allocation is not implemented.

agents have the same consumption in period $t+1$; i.e. $\hat{\sigma}_{t+1}(\theta^T)=\theta_{t+1}$ and $c^{\hat{\sigma}}(\theta^{t+1})=c(\theta^{t+1})$. This indicates that the discounted, post-tax marginal utility of consumption in the right-hand side of Euler equation (16a) is the same for both shirking and truth-telling agents. However, by deviating in period t, the shirking agent can consume more than a truth-telling agent, so the marginal utility of consumption in period t in the left-hand side of (16a) for a shirking agent is different from that of a truth-telling agent. As such, there is no capital tax rate $\tilde{\tau}_k(\hat{\theta}^t)$ that satisfies Euler equation (16a) for both shirking and truth-telling agents.

To resolve this problem, the income condition (15b) is needed in our tax system. If we denote $\eta^{\hat{\sigma}}(\theta^t)$ as the multiplier of the income condition (15b) in the agents' problem, then the Euler equation (16a) is revised as follows.

$$u'(\tilde{c}_t)=\beta R_{t+1}\left(1-\tilde{\tau}_k(\hat{\theta}^t)\right)E_t\left[u'(\tilde{c}_{t+1})\right]-\eta^{\hat{\sigma}}\left(\theta^t\right). \tag{16b}$$

Agents of different types have different shadow prices $\eta^{\hat{\sigma}}(\theta^t)$. Moreover, as the linear labor income tax rate $\tilde{\tau}_z(\hat{\theta}^t)$ directly affects the income condition (15b), the shadow price of the income condition $\eta^{\hat{\sigma}}(\theta^t)$ changes with different values of $\tilde{\tau}_z(\hat{\theta}^t)$. Hence, this income condition makes room for the constrained efficient allocation to satisfy (16b) for both shirking and truth-telling agents. In other words, different multipliers of the income conditions make it possible to find linear income tax rates $(\tilde{\tau}_k(\hat{\theta}^t),\tilde{\tau}_z(\hat{\theta}^t))$ so constrained efficient allocations resulting from reporting strategies specified by shirking and truth-telling agents satisfy (16b).

Our implementation of the constrained efficient allocations in terms of linear capital and labor income taxes is different from the implementation in terms of a linear capital income tax in Grochulski and Piskorski (2010). To avoid an agent from deviating from labor and saving jointly, these two authors used a non-linear labor income tax to restrict the agent's labor to the constrained efficient level. Under the restriction, however, their linear capital income tax-adjusted Euler equations associated with truth-telling and shirking strategies can be consistent with each other only if there exists a deferred capital tax.

By contrast, we do not restrict agents' labor choices. Instead, we only impose an income condition $S(\tilde{x}^t,\tilde{z}_t,\tilde{k}_{t+1})=0$ to restrict the post-tax total income to be consistent with the constrained efficient level. Our tax system has two merits. First, it allows agents to choose their labor and savings jointly without the concern of a double deviation problem, provided that agents' post-tax income is consistent with the constrained efficient level. Second, a deferred capital tax is not necessary, because our linear labor income tax rate can function like the deferred capital tax in Grochulski and Piskorski (2010). Our linear labor income taxes play a role that makes our linear capital income tax-adjusted Euler equations associated with truth-telling and shirking strategies consistent with each other.

6. Numerical Analysis

In this section, we offer numerical analysis to highlight the quantitative importance of our results. Our numerical analysis takes a middle position between a simple demonstration of the optimal mechanism and a careful calibration of quantitative implications for the wedge. The numerical analysis has four goals: firstly, to demonstrate the average capital and labor wedge over time; secondly, to illustrate the capital and the labor wedge for different skill types in some working periods; thirdly, to exhibit whether the capital and the labor wedge are progressive or regressive in agents' types; fourthly, to highlight the redistribution effect in terms of the welfare gain and compare our history-dependent tax system with a simple history-independent, non-linear tax system.

6.1 Calibration

We calibrate our model economy based on the US data and then quantitatively solve the constrained efficient allocation. The calibration proceeds as follows.

Firstly, we construct a baseline decentralized economy, with a linear income tax system with tax rates being set to current average levels in the US. The structure of our baseline economy is the same as the model in Section 4 except for no social planner.

Agents are set to live 60 years, working for 40 years and then retiring for 20 years. In the baseline, some parameter values are set exogenously, based on the existing literature, normalization or assumptions. Table 1 lists all parameter values except two. Following Stantcheva (2017), these two parameter values are endogenously calibrated to match the moments from the data, targeting the wage premium and the ratio of human capital expenses to lifetime income. Table 2 lists these two calibrated parameter values.

[Insert Tables 1 and 2 here]

For the tax system, according to McDaniel (2007), average capital and labor income tax rates in the US during 1960-2007 are around 0.3 and 0.2, respectively. Thus, in our baseline economy, we set $\tau_k^b = 30\%$ and $\tau_z^b = 20\%$. We assume zero government expenditure $G_t = 0$, so the tax revenue is equally redistributed to agents as a lump-sum transfer LS_t. An agent's budget constraints are as follows.

$$c_t + x_t + y_t + k_{t+1} \leq \left(1-\tau_z^b\right) w_t z_t + \left(1-\tau_k^b\right) R_t k_t + LS_t. \tag{17a}$$

The periodic utility function during working years takes the following form.

$$u(c_t) - \phi\left(\frac{z_t}{h_t}\right) = \log(c_t) - \frac{1}{\kappa}\left(\frac{z_t}{h_t}\right)^{\kappa}.$$

Following Farhi and Werning (2013), we set $\kappa = 3$, which implies the Frisch elasticity for labor of 0.5. The discount factor is set at 5 per annum, which gives $\beta = 0.95$. Also, we set an equal discount factor for agents and the planner, which implies that $1/R_t = \beta$, and wage rates are normalized to 1.

As for the human capital accumulation, under construction, the initial human capital level is equal for all agents; hence, the initial human capital level is normalized to $h_1 = 1$. The human capital accumulation is (9), in which next period's human capital depends on the function $\psi(x, y)$ and agents' types θ_t. Following Ewijk and Tang (2000), we use the Cobb-Douglas form for the function. Therefore, the level of agent's human capital in periods $t = 2, \ldots, 40$ takes the following form:

$$h_t = B\left(x_{t-1}^{1-\rho} y_{t-1}^{\rho}\right)^{\eta} + \theta_{t-1}. \tag{17b}$$

Note that the form reduces to the case with only verifiable education expenses if $\rho = 0$. For parameter values, following Ewijk and Tang (2000), we set $\rho = 0.667$, and the technology level B is normalized to 1. The parameter η is calibrated to match the targeted ratio of the net present value of lifetime education expenses over the net present value of lifetime income. Stantcheva (2017) computed and found the ratio of 19 percent, with which we go along.

Moreover, following Farhi and Werning (2013), the skill shock θ is an AR(1) process with white noise, where the white noise is interpreted as measurement error with the coefficient of auto-correlation being very close to one. A geometric random walk is adopted as follows.

$$\theta_t = \varepsilon_t \theta_{t-1} \text{ with } \quad \log \varepsilon_t \overset{i.i.d}{\sim} N\left(-\tfrac{\hat{\sigma}^2}{2}, \hat{\sigma}^2\right),$$

wherein the distribution moves proportionally over a finite interval $[\underline{\theta}, \overline{\theta}]$ with the value of the degree of uncertainty at $\hat{\sigma}^2 = 0.0095$.[42] The lower bound of shocks is normalized to $\underline{\theta} = 0.5$, and the upper bound $\overline{\theta}$ is calibrated to match the wage premium.

The estimated value for the wage premium in the literature lies within 1.2 and 2.4, as the estimated range is 1.26-1.74 in Murphy and Welch (1992), 1.37-1.75 in Autor et al. (1998), 1.7-2.4 in Heathcote et al. (2005), and 1.2-2.2 in James (2012). Our calibration targets a medium value of 1.8. To match the wage premium, we go along Stantcheva (2017) and compute the labor income of the top 42 percent relative to the bottom 42 percent in the population. We calibrate the values of η and $\overline{\theta}$ in the following way.

Firstly, we derive the individual's problem in the decentralized baseline economy. In Appendix A.11, we have set up the problem of an agent with skill type θ^t. Based on the problem's first-order conditions, we use the parametric functional forms to simplify these conditions to the following four equations.

$$\left(c_t\right)^{-\chi} = \beta R_{t+1}\left(1-\tau_k\right) E_t\left[\left(c_{t+1}\right)^{-\chi}\right],$$

$$\left(1-\tau_z\right) w_t \left(c_t\right)^{-\chi} = \left(\tfrac{z_t}{h_t}\right)^{\gamma-1} \tfrac{1}{h_t},$$

$$\left(c_t\right)^{-\chi} = \beta E_t\left[\left(\tfrac{z_{t+1}}{h_{t+1}}\right)^{\gamma}\right] \frac{B\eta(1-\rho)(x_t)^{\eta(1-\rho)-1}(y_t)^{\eta\rho}}{h_{t+1}},$$

$$\left(c_t\right)^{-\chi} = \beta E_t\left[\left(\tfrac{z_{t+1}}{h_{t+1}}\right)^{\gamma}\right] \frac{B\eta\rho(x_t)^{\eta(1-\rho)}(y_t)^{\eta\rho-1}}{h_{t+1}}.$$

Next, with these four equations above and (17a)-(17b), there are six equations. Based on the parameter values in Table 1, we solve the allocation $\{c_t(\theta^t), z_t(\theta^t), x_t(\theta^t), y_t(\theta^t), k_{t+1}(\theta^t), h_t(\theta^t)\}$ for each skill type history θ^t. Specifically, we guess initial values for η and $\overline{\theta}$, and use these six equations

[42] The degree of uncertainty $\hat{\sigma}^2$ is empirically estimated by matching the increase in the cross-sectional variance of wages or earnings in a given cohort as this cohort ages. The estimate depends on whether time fixed effects (smaller estimates) or cohort fixed effects (larger estimates) are imposed, and on the time period (larger estimates in the 1980's). Using cohort fixed effects over the period 1967–1996, Heathcote *et al.* (2005) find $\hat{\sigma}^2 = 0.0095$ for the wage of male individuals, which was used by Farhi and Werning (2013), and we follow suit.

to solve the allocation for each skill type history θ^t. The resulting allocation is then used to compute the wage premium and the ratio of education expenses to income. If the resulting wage premium and the ratio of education expenses to income are different from the target values of 1.8 and 0.19, respectively, we adjust the values of η and $\overline{\theta}$ and re-compute the allocation using these six equations. Then again, we compute the resulting wage premium and the ratio of education expenses to income. The process is repeated, until the wage premium and the ratio of education expenses to income reach their target values. The resulting calibrated value is $\eta = 0.4$ and $\overline{\theta} = 1.5$. See Table 2. We are ready to envisage the simulation results.

6.2 Simulation Results

We apply these parameter values from calibration to the second-best economy and calculate the policy functions with respect to the constrained efficient allocation of each type. Using the computed policy functions, we carry out Monte Carlo simulations with one million agents evolving through periods $t = 1,2,\ldots,T$. Note that agents do not work but consume the same after retirement, so all the wedges are zero after retirement. Therefore, we only focus on working periods $t = 1,2,\ldots,40$ in this subsection.[43]

In the first period, we normalize $v_1 = 5$ and $h_1 = 1$ and solve the cost minimization problem (25b) in Appendix A.8, which is rewritten as follows.

$$\mathcal{K}(v_1 = 5, h_1 = 1, t = 1) = \min \int \left[c(\theta) + x(\theta) + y(\theta) - w_1 z(\theta) + \tfrac{1}{R_2} \mathcal{K}\left(v(\theta), \Delta(\theta), h(\theta), \theta, 2\right) \right] \pi(\theta) d\theta,$$

In later periods $t = 2,\ldots,T,$ the state variables $v(\theta),\ \Delta(\theta),\ h(\theta)$ are solved by the problem in the previous period, and then the policy functions are solved by using the cost minimization problem (25a) in Appendix A.8. To highlight the role of privately observed human capital investment, we also simulate an otherwise the same model as our model except that both $x(\theta)$ and $y(\theta)$ are observable. This enables us to compare simulation results of our model with those of the model without privately observed human capital investment. An otherwise our model except for $y(\theta)$ being observable, is dubbed no private HCI.

6.2.1 Capital wedge

First, the average capital wedge over time is demonstrated in Figure 1. As the figure shows, the average capital wedge is positive and decreases over time throughout all working periods in both our model (the solid line) and the model without unobservable human capital investment (the dotted line, labelled no private HCI). When comparing our model with the model of no private HCI, the capital

[43] The solution of the relaxed planning problem may not be the solution of the original social planning problem. Following Farhi and Werning (2013) and Stantcheva (2017), we have verified that our solution satisfies the IC constraint in Subsection 6.4. For agents of all skill types, if they truly report the types, they obtain the highest lifetime utility. Therefore, the solution that we characterize in section 4 is indeed the solution of the original social planning problem.

wedge is higher in our model. This result comes from the *HCI effect* induced by unobservable human capital investment.

[Insert Figure 1 here]

According to Proposition 7, our capital wedge involves three effects as follows:

$$\tau_{k_t}\left(\theta^t\right) = \{\textit{insurance effect}\} + \{\textit{ current period's HCI effect}\} + \{\textit{ next period's HCI effect}\}.$$

We decompose the average capital wedge over time of our model in Figure 1 into these three effects with the results illustrated in Figure 2. As is standard, the insurance effect is unambiguously positive, which is the average capital wedge in the model of no private HCI in Figure 1. Moreover, Figure 2 indicates that the current period's HCI effect is positive, while the next period's HCI effect is negative. The simulation result shows that the magnitudes of these effects are diminishing over time, but the positive current period's HCI effect quantitatively dominates the negative next period's HCI effect. As a result, the capital wedge in our model is higher than the model of no private HCI in Figure 1.

[Insert Figure 2 here]

To understand the distribution of the capital wedge across different skill types, we simulate the scatter plot of the capital wedge against contemporary skill types in periods 1, 2,…, 40 over the life cycle. To save space, Figure 3 presents the scatter plot in the mid-working period at $t = 20$ as an example. As is clear, from Figure 3(a), the capital wedge is not only positive but also progressive against contemporaneous skill types, with a higher capital wedge for a higher-skill type. In Figure 3 (b)-(d), we decompose the capital wedge against skill types into three sources. While Figure 3(b) is the insurance effect, which is positive and regressive against skill types, Figure 3(c) is the current period's HCI effect, which is positive and progressive against skill types, and Figure 3(d) is the next period's HCI effect, which is negative and diminishing in skill types.

[Insert Figure 3 here]

Due to the insurance effect, the existing dynamic Mirrlees literature has obtained the capital wedge that is positive, as illustrated in Figure 3(b). As the insurance effect diminishes over time, the average capital wedge monotonically decreases and approaches to zero when nearing retirement (e.g., Farhi and Werning, 2013; Stantcheva, 2017). By contrast, the current period's HCI effect and the next period's HCI effect are both at work in our model. While the next period's negative HCI effect and the positive insurance effect are both weak with a factor of 10^{-4}, the current period's positive HCI effect is strong with a factor of 10^{-3}. As a result, the average capital wedge is positive for all the working periods. Moreover, due to the large current period's HCI effect, the capital wedge is positive and progressive against contemporary skill types. To the best of our knowledge, these are new results.

6.2.2 Labor wedge

Next, we simulate the labor wedge in the periods 1, 2…, 40 over the life cycle. As seen in Figure 4, the average labor wedge is negative early and positive later in working periods in our model (the solid line), as compared to being always positive in the model with no private HCI (the dotted line).

[Insert Figure 4 here]

According to Proposition 8, the labor wedge in our model involves two different effects as follows:

$$\tau_{z_t}\left(\theta^t\right) = \{\textit{shirking - preventing effect}\} + \{\textit{skill - fostering effect}\}.$$

Figure 5 decomposes the labor wedge over the life cycle in our model into two sources. As can be seen, the shirking-preventing effect has a positive impact on the labor wedge, while the skill-fostering effect has a negative effect on the labor wedge. Since the negative skill-fostering effect quantitatively dominates the positive shirking-preventing effect in earlier periods, the average labor wedge is negative in earlier periods. Moreover, Figure 5 indicates that the shirking-preventing effect is increasing over time, while the skill-fostering effect is diminishing to zero when nearing retirement. As a result, the shirking-preventing effect accounts for all effects on the labor wedge when nearing retirement.

[Insert Figure 5 here]

To understand the distribution of the labor wedge, we simulate the scatter plot of the labor wedge against skill types in periods 1, 2,…, 40 over the life cycle. To save space, Figure 6 reports the scatter plot using the mid-working period at $t = 20$ as an example. In Figure 6(a), the labor wedge is hump-shaped against contemporary skill types, just like that in the Mirrlees model in Golosov et al (2006) and Ales et al (2015). Yet, because of the skill-fostering effect, our labor wedge is negative at the top and the bottom of the skill distribution, departing from the standard zero-tax result in the existing Mirrlees literature. When decomposing the labor wedge in Figure 6(a) into the two sources, while the shirking-preventing effect in Figure 6(b) is also hump-shaped and positive against skill types, the skill-fostering effect in Figure 6(c) is negative with the magnitude decreasing in skill types. At the bottom of the skill distribution, the negative skill-fostering effect quantitatively dominates the positive shirking-preventing effect. At the top of the skill distribution, although the negative skill-fostering effect is small, the positive shirking-preventing effect is so small that is quantitatively dominated by the negative skill-fostering effect. As a result, the labor wedge is negative at the bottom and the top of the skill distribution.

[Insert Figure 6 here]

In the existing Mirrlees literature with exogenous skills, because of the shirking-preventing effect, a positive labor wedge is designed for the redistribution purpose to prevent skilled agents from shirking. Farhi and Werning (2013) studied a model with exogenous skills evolving according to a stochastic AR(1) process. Their results indicate that the labor wedge is positive and regressive against contemporary skill types. Stantcheva (2017) analyzed a model with endogenous skills via verifiable education expenses. Her quantitative results suggest that the labor wedge may be regressive or progressive, depending on whether the Hicksian coefficient of complementarity between skill types and human capital is larger or smaller than 1. Different from these two papers, with unobservable human capital investment in our model, the labor wedge is determined by the interaction of the positive shirking-preventing effect and the negative skill-fostering effect. Thus, the labor wedge can be positive or negative, depending on the magnitude of these two effects. In early periods, the negative skill-fostering effect dominates the positive shirking-preventing effect, so the labor wedge is negative. Moreover, in a middle working period, because the skill-fostering effect dominates the shirking-preventing effect, the labor wedge is not zero but negative at the bottom and the top of the skill distribution. These are new results.

6.3 Welfare Gains and Simple History-independent Policy

In this subsection, we answer the following two questions. First, comparing with the laissez-faire economy without taxes, what is the welfare gain of the constrained efficient allocation in our second-best planning economy? Second, if our history-dependent tax system is too complicated to be feasible, how well can a simple history-independent tax policy do in our model?

To answer the first question, we compare the welfare gain of our second-best planning economy to the laissez-faire economy without taxes. Let $W^{LF}(c_t^{LF}(\theta), l_t^{LF}(\theta))$ be the welfare of the laissez-faire economy without taxes (*LF*), where $c_t^{LF}(\theta)$ and $l_t^{LF}(\theta)$ are, respectively, consumption and the labor supply of type θ in time t. Let W^{SB} be the welfare of our second-best planning economy (*SB*). The welfare gain of the second-best planning economy from the laissez-faire economy without taxes is defined in terms of consumption equivalence: the percentage increase in consumption in the second-best economy relative to the laissez-faire economy without taxes. Let ω denote the percentage increase in consumption. Then, the following condition is met.

$$W^{LF}\left((1+\omega_{SB})c_t^{LF}(\theta), l_t^{LF}(\theta)\right) = W^{SB}.$$

In Farhi and Werning (2013), they compare the welfare gain with respect to three different estimated values of skill risks $\hat{\sigma}^2$: a low risk with $\hat{\sigma}^2 = 0.00625$, a medium risk with $\hat{\sigma}^2 = 0.0095$, and a high risk with $\hat{\sigma}^2 = 0.0161$. Following their work, we also compute the welfare gain with respect to these three different values of skill risks. The results are in the top row of Table 3. The welfare gains are all positive. Moreover, the welfare gain is increasing with the value of skill risks. Intuitively, the higher the skill risk, the higher is the welfare gain in our second-best economy.

[Insert Table 3 here]

Next, our second-best economy requires a *history-dependent* tax system. Yet, if our history-dependent tax system is too complex and infeasible, how well can a simple *history-independent* tax policy do? We compute the welfare gain of our model under a simple history-independent tax policy.

To this end, we consider a nonlinear tax and transfer policy defined by the following function.

$$T(z) = z - (1-\tau)z^{1-\hat{\lambda}}, \tag{18}$$

where z is income, τ is the tax rate, and $\hat{\lambda}$ is the degree of progressivity of the tax policy. This specification is well-known in public finance.[44] We choose this simple nonlinear tax function, since when

[44] The function was introduced by Feldstein (1969). More recently, Benabou (2000) and Heathcote et al. (2017) have applied this policy into dynamic macroeconomic models with heterogeneous agents.

the income is below $z_0 = (1-\tau)^{\frac{1}{\hat{\lambda}}}$, the tax is negative, which captures negative labor wedges for lower type agents in early periods.

Following Heathcote et al. (2017), the value of $\hat{\lambda}$ is set to be 0.181 to match with the estimation value. Given $\hat{\lambda}$, the second parameter τ is chosen to generate a welfare gain closest to our history-dependent second-best model, which is $\tau = 0.2$.

The welfare gain of our model under the simple history-independent policy (18) for different skill risks is illustrated in the second row of Table 3. It is clear that, in terms of the overall welfare gain, the simple non-linear tax is close to our second-best planning economy. (See the third row.) This result lends supports to the argument made by Heathcote et al. (2017, p.1697), that the parametric tax specification (18) is sufficiently flexible that the welfare gain of moving from the simple history-independent specification to a constrained-efficient Mirrleesian tax schedule is likely to be small.

6.4 Ex-post ICC verification

Our model studies the relaxed problem based on the first-order approach, which replaces the incentive compatible conditions with the envelope condition. In this subsection, we numerically verify whether the solution to the relaxed planning problem is the solution to the full program. In other words, we numerically verify whether the envelope condition used in our model solves for the allocation that implies *ex post* incentive compatibility.

We simulate the utility gains of different reporting strategies in our model. Specifically, we simulate the utility of our model in all periods $t = 1,2,\ldots,40$ over the life cycle, and then calculate the difference $W^{\sigma}(\theta) - W(\theta)$, defined in (10a)-(10b), that is, the utility gains of reporting strategies σ from the truth-telling strategy in the model. We do the simulation based on the first-best allocation (without the envelope condition) and constrained efficient allocation (with the envelope condition), respectively. To save space, Figure 7 and Figure 8 only demonstrates the utility gains for period $t = 20$ under the state variables $\mathcal{K}(v = 3.26, \Delta = 0.45, h = 1.17, \theta\_ = 1, t = 20)$ as an example of ex post verification, where the x-axis is the true type θ, the y-axis is the reporting type σ, and the z-axis is utility gains of reporting types σ from the truth-telling strategy θ. As can be seen from Figure 7, which is based on the first-best allocation without the envelope condition, agents can obtain a higher utility by under-reporting their types, while from Figure 8, which is based on the constrained efficient allocation with the envelope condition, agents cannot obtain a higher utility from any misreporting strategy. The truth-telling strategy always gives the highest utility in our model (cf. the solid diagonal bold line). The comparison between Figure 7 and Figure 8 indicates that the envelope condition indeed generates the incentive-compatible allocation in our model and thus, the validity of the relaxed planning problem is guaranteed.

[Insert Figures 7 and 8 here]

7. Concluding Remarks

This paper studies wedges and the tax implementation in a dynamic Mirrlees economy. Our paper adds unobservable human capital investment into the existing model with observable human capital investment over the life cycle. In the model, in addition to working and savings, agents choose expenses for consumption and education over time. The key feature is indistinguishable consumption from education expenses. The social planner chooses constrained efficient allocations that maximize the utilitarian social welfare subject to resource constraints and incentive-compatibility constraints. We characterize capital wedge, labor wedge and net human capital wedge in the resulting constrained efficient allocations, and then construct a system of linear capital and labor income taxes to implement constrained efficient allocations in a decentralized economy.

We obtain two results. First, the capital wedge is positive and remains so even if there is no uncertainty of skill shocks and thus, no insurance purposes. Moreover, the labor wedge is negative early and positive later in the life cycle. These wedges emerge from distortions to consumption due to indistinguishable consumption from education expenses. Our positive capital wedge arises not only from the standard insurance effect but also from the new human capital investment effect due to indistinguishable consumption and education expenditure. Our negative labor wedge early in the life cycle arises, because the new skill-fostering effect dominates the standard shirk-preventing effect.

Second, in order to implement the constrained efficient allocations as an outcome in a decentralized economy, we construct a tax system with linear capital and labor income tax rates. By imposing an income condition into our tax system, agents can jointly choose labor and savings, and deferred capital taxes are not necessary in our framework.

Finally, our simulation results suggest that the average capital wedge is positive and decreasing over time, and, due to the human capital investment effect, is higher than the model without private human capital investment. Even in the mid-working period, the capital wedge is positive and progressive against skill types. Moreover, the average labor wedge is negative in early periods, increases over time, and is positive in later periods. In a mid-working period, the labor wedge is hump-shaped against skill types with negative labor wedges at the top and the bottom of the productivity distribution, different from the standard zero-tax result at top and the bottom of the productivity distribution. Further, our history-dependent optimal tax policy results in a large welfare gain when comparing with a laissez-faire economy without taxes, and if our history-dependent policy is too complicated to be feasible, a simple history-independent non-linear tax policy would give a close welfare gain.

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Appendix

This Appendix offers proofs for the lemmas and propositions in the text.

A.1 Proof of Proposition 1 in Subsection 2.3

Suppose that the constrained efficient allocation $\{c_1^\sigma(\theta), y_1^\sigma(\theta), h_2^\sigma(\theta)\}$ is the solution for the problem in (2). Then, by using the constraints in (2), we can replace $c_1^\sigma(\theta)$ by $c_1(\sigma)+y_1(\sigma)-y_1^\sigma(\theta)$ and $h_2^\sigma(\theta)$ by $(1-\delta_h)h_1+\psi(x_1(\sigma), y_1^\sigma(\theta))+\theta$. Then, the problem in (2) becomes:

$$\max_{y_1^\sigma(\theta)} u\big(c_1(\sigma)+y_1(\sigma)-y_1^\sigma(\theta)\big)-\phi\left(\frac{z_1(\sigma)}{h_1}\right)+\beta\left[u\big(c_2(\sigma)\big)-\phi\left(\frac{z_2(\sigma)}{(1-\delta_h)h_1+\psi\big(x_1(\sigma),y_1^\sigma(\theta)\big)+\theta}\right)\right]. \tag{19a}$$

We denote $\phi_h\left(\frac{z_t}{h_t}\right)\equiv -\phi'\left(\frac{z_t}{h_t}\right)\frac{z_t}{(h_t)^2}<0$ and $\phi_{hh}\left(\frac{z_t}{h_t}\right)\equiv \phi''\left(\frac{z_t}{h_t}\right)\frac{(z_t)^2}{(h_t)^4}+2\phi'\left(\frac{z_t}{h_t}\right)\frac{z_t}{(h_t)^3}>0$. Then, the first-order condition of the problem (19a) is

$$u'\big(c_1(\sigma)+y_1(\sigma)-y_1^\sigma(\theta)\big)+\beta\phi_h\left(\frac{z_2(\sigma)}{(1-\delta_h)h_1+\psi\big(x_1(\sigma),y_1^\sigma(\theta)\big)+\theta}\right)\psi_y\big(x_1(\sigma),y_1^\sigma(\theta)\big)=0, \tag{19b}$$

which is the first part of Proposition 1.

Moreover, differentiating (19b) with respect to θ yields

$$u''(c_1^\sigma(\theta))\tfrac{\partial}{\partial\theta}y_1^\sigma(\theta)=\beta\left[\phi_{hh}\left(\tfrac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\left[\psi_y\big(x_1(\sigma),y_1^\sigma(\theta)\big)\tfrac{\partial}{\partial\theta}y_1^\sigma(\theta)+1\right]\psi_y\big(x_1(\sigma),y_1^\sigma(\theta)\big)+\phi_h\left(\tfrac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\psi_{yy}\big(x_1(\sigma),y_1^\sigma(\theta)\big)\tfrac{\partial}{\partial\theta}y_1^\sigma(\theta)\right],$$

which gives

$$\tfrac{\partial}{\partial\theta}y_1^\sigma(\theta)=\frac{\beta\phi_{hh}\left(\tfrac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\psi_y\big(x_1(\sigma),y_1^\sigma(\theta)\big)}{u''\big(c_1^\sigma(\theta)\big)-\beta\left[\phi_{hh}\left(\tfrac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\big(\psi_y\big(x_1(\sigma),y_1^\sigma(\theta)\big)\big)^2+\phi_h\left(\tfrac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\psi_{yy}\big(x_1(\sigma),y_1^\sigma(\theta)\big)\right]}$$

$$\frac{\partial}{\partial\theta}h_2^\sigma(\theta)=\psi_y\left(x_1(\sigma),y_1^\sigma(\theta)\right)\frac{\partial}{\partial\theta}y_1^\sigma(\theta)+1$$
$$=\frac{u''\left(c_1^\sigma(\theta)\right)-\beta\phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\psi_{yy}\left(x_1(\sigma),y_1^\sigma(\theta)\right)}{u''\left(c_1^\sigma(\theta)\right)-\beta\left[\phi_{hh}\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\left(\psi_y\left(x_1(\sigma),y_1^\sigma(\theta)\right)\right)^2+\phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\psi_{yy}\left(x_1(\sigma),y_1^\sigma(\theta)\right)\right]}. \tag{19d}$$

As $u''<0$, $\psi_{yy}<0$, $\phi_{hh}>0$, $\phi_h<0$ and $\psi_y>0$, (19c) and (19d) imply $\frac{\partial}{\partial\theta}y_1^\sigma(\theta)<0$ and $\frac{\partial}{\partial\theta}h_2^\sigma(\theta)>0$. □

A.2 The relaxed planning problem and the first-order conditions in Subsection 3.1.

Let λ_t be the shadow price of the resource constraint in period $t=1,2$ and $\mu(\theta)$ be the co-state variable associated with $\dot{W}(\theta)$ Moreover, we use (4c) to replace $c_2(\theta)$ by

$$u^{-1}\left\{\frac{1}{\beta}\left[W(\theta)+\phi\left(l_1(\theta)\right)-u\left(c_1(\theta)\right)\right]+\phi\left(l_2(\theta)\right)\right\}$$

Then, the Hamiltonian of the relaxed planning problem is given by

$$\begin{aligned}\mathcal{H}&=\pi(\theta)W(\theta)\\&+\lambda_1\left[F\left(K_1,\int_\Theta\pi(\theta)z_1(\theta)d\theta\right)+(1-\delta_k)K_1-G_1-\pi(\theta)c_1(\theta)-\pi(\theta)y_1(\theta)-\pi(\theta)x_1(\theta)-K_2\right]\\&+\lambda_2\left\{F\left(K_2,\int_\Theta\pi(\theta)z_2(\theta)d\theta\right)+(1-\delta_k)K_2-G_2-\pi(\theta)u^{-1}\left[\frac{1}{\beta}\left(W(\theta)+\phi\left(\frac{z_1(\theta)}{h_1}\right)-u\left(c_1(\theta)\right)\right)+\phi\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\right\}\\&+\mu(\theta)\left\{-u'\left(c_1(\theta)\right)\frac{\partial y_1(\theta)}{\partial\theta}+\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{\left[h_2(\theta)\right]^2}\frac{\partial h_2(\theta)}{\partial\theta}\right\},\end{aligned}$$

The first-order conditions with respect to $c_1(\theta), z_1(\theta), z_2(\theta), K_2$ and $x_1(\theta)$ are as follows.

$$\frac{\partial\mathcal{H}}{\partial c_1(\theta)}=-\lambda_1\pi(\theta)+\lambda_2\pi(\theta)\frac{u'\left(c_1(\theta)\right)}{\beta u'\left(c_2(\theta)\right)}-\mu(\theta)u''\left(c_1(\theta)\right)\frac{\partial y_1(\theta)}{\partial\theta}=0, \tag{20a}$$

$$\frac{\partial\mathcal{H}}{\partial z_1(\theta)}=\lambda_1F_z\left(K_1,Z_1\right)\pi(\theta)-\lambda_2\pi(\theta)\frac{\phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{h_1}}{\beta u'\left(c_2(\theta)\right)}=0, \tag{20b}$$

$$\frac{\partial\mathcal{H}}{\partial z_2(\theta)}=\lambda_2\pi(\theta)\left[F_z\left(K_2,Z_2\right)-\frac{\phi'\left(\frac{z_2(\theta)}{h_2}\right)\frac{1}{h_2}}{u'\left(c_2(\theta)\right)}\right]+\frac{\mu(\theta)\beta\left[\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)}+\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\frac{\partial h_2(\theta)}{\partial\theta}}{\left[h_2(\theta)\right]^2}=0,\ldots\ldots. \tag{20c}$$

$$\frac{\partial\mathcal{H}}{\partial x_1(\theta)}=-\lambda_1\pi(\theta)+\frac{\lambda_2\pi(\theta)\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{u'\left(c_2(\theta)\right)\left[h_2(\theta)\right]^2}-\frac{\beta\mu(\theta)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\left[h_2(\theta)\right]^3}\left\{\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)}+2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}=0, \tag{20d}$$

$$\frac{\partial\mathcal{H}}{\partial K_2}=-\lambda_1+\lambda_2\left[F_k\left(K_2,Z_2\right)+1-\delta_k\right]=0. \tag{20e}$$

Note that the choice of capital in the last condition is the same as the corresponding condition in the Ramsey model. Moreover, if the IC constraint is not binding and thus $\mu(\theta) = 0$, the first 4 conditions reduce to standard conditions for consumption, effective labor and education investment in the Ramsey model, wherein the discounted marginal utility of consumption and effective labor for each type is equal to the marginal cost. However, if the IC constraint binds and thus $\mu(\theta) \neq 0$, these conditions differ from those in the Ramsey model.

First, the equation (6a) can be derived by using (20a) and (20e). Second, using (20a) and (20b), we have following equation

$$\frac{\phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{h_1}}{u'\left(c_1(\theta)\right)}\left(1-\beta u'\left(c_2(\theta)\right)\frac{\mu(\theta)u''\left(c_1(\theta)\right)\frac{\partial y_1(\theta)}{\partial\theta}}{\lambda_2\pi(\theta)u'\left(c_1(\theta)\right)}\right)^{-1} = F_z\left(K_1, Z_1\right). \tag{21a}$$

Based on (20b) and (20e), we have

$$\beta\left[F_k\left(K_2, Z_2\right)+1-\delta_k\right]u'\left(c_2(\theta)\right) = \phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{F_z(K_1, Z_1)h_1}. \tag{21b}$$

The equation (6b) can be derived by using (21a), (21b) and (20e). Third, the equation (6c) is derived by using (20c). Finally, using equation (20e), the equation (20d) becomes

$$1 = \frac{\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)\psi_x\left(x_1(\theta), y_1(\theta)\right)}{\left[F_k(K_2, Z_2)+1-\delta_k\right]u'\left(c_2(\theta)\right)\left[h_2(\theta)\right]^2} - \frac{\beta\mu(\theta)z_2(\theta)\psi_x\left(x_1(\theta), y_1(\theta)\right)}{\lambda_1\pi(\theta)\left[h_2(\theta)\right]^3}\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)}+2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}. \tag{21c}$$

Then, (6d) is derived from multiplying $u'\left(c_1(\theta)\right)$ on the both sides of (21c).

A.3 Proof of Lemma 1 in Subsection 3.1

This proof is close to that in Stantcheva (2017). With boundary conditions $\mu(\overline{\theta}) = \mu(\underline{\theta}) = 0$ in (5a), according to Rolle's theorem, there exists $\theta' \in (\underline{\theta}, \overline{\theta})$ such that $\frac{d}{d\theta}\mu(\theta') = 0$. The law of motion (5b) gives

$$\lambda_2 = \beta u'\left(c_2(\theta')\right) > 0.$$

Because $c_2(\theta)$ is monotone increasing in θ, the above equation implies that $\lambda_2 < \beta u'(c_2(\theta))$ for $\theta < \theta'$ and $\lambda_2 > \beta u'(c_2(\theta))$ for $\theta > \theta'$ According to the law of motion in (5b), this implies that $\frac{d}{d\theta}\mu(\theta) < 0$ for $\theta < \theta'$ and $\frac{d}{d\theta}\mu(\theta) > 0$ for $\theta > \theta'$ That is, the derivative of $\mu(\theta)$ is negative for small values of θ and positive for large values of θ. With $\mu(\overline{\theta}) = \mu(\underline{\theta}) = 0$ this ensures that $\mu(\theta)$ is negative for $\theta \in (\underline{\theta}, \overline{\theta})$, as illustrated in Figure 5.

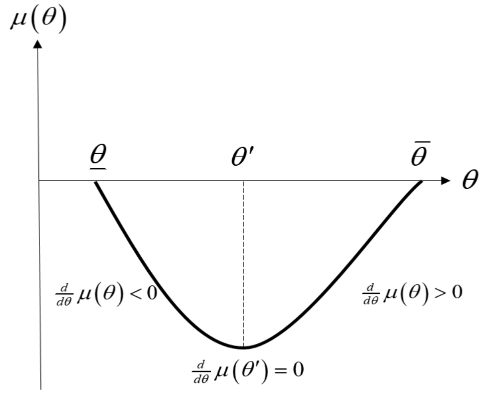


Figure 5. Non-positive co-state $\boldsymbol{\mu(\theta)}$

Alternatively, if we integrate (5b) and use the boundary condition, $\mu(\bar{\theta})=0$, we obtain

$$\mu(\theta)=\int_{\theta}^{\bar{\theta}}\left(1-\frac{\lambda_2}{\beta u'(c_2(\theta''))}\right)\pi(\theta'')d\theta''.$$

Note that, for $\theta\in[\theta',\bar{\theta})$, due to the fact that $\lambda_2>\beta u'(c_2(\theta))$ for $\theta>\theta'$ the above equation integrates over non-positive variables only. Thus, the above equation implies $\mu(\theta)<0$ for $\theta\in[\theta',\bar{\theta})$.

Similarly, integrating (5b) and using the boundary condition $\mu(\underline{\theta})=0$ yields

$$\mu(\theta)=\int_{\underline{\theta}}^{\theta}\left(-1+\frac{\lambda_2}{\beta u'(c_2(\theta''))}\right)\pi(\theta'')d\theta''.$$

Since $\lambda_2<\beta u'(c_2(\theta))$ for $\theta\in(\underline{\theta},\theta']$, the above equation integrates over non-negative variables only when $\theta<\theta'$, which also implies $\mu(\theta)<0$ for any $\theta\in(\underline{\theta},\theta']$.

Hence, we obtain that $\mu(\theta)<0$ for any $\theta\in(\underline{\theta},\bar{\theta})$. □

A.4 Proofs of Propositions 2-4 and Corollaries 1-2 in Subsection 3.2

Proof of Proposition 2:

Based on equation (6a) and (20e), we have

$$\frac{1}{u'(c_1(\theta))}=\frac{1}{\beta R_2\,u'(c_2(\theta))}-\frac{\mu(\theta)u''\big(c_1(\theta)\big)}{\lambda_1\pi(\theta)u'\big(c_1(\theta)\big)}\frac{\partial y_1(\theta)}{\partial\theta}.$$

According to Lemma 1 and Proposition 1 (with $\sigma=\theta$), $\lambda_2>0$, $\mu(\theta)<0$ and $\frac{\partial y_1(\theta)}{\partial\theta}<0$ for any $\theta\in(\underline{\theta},\bar{\theta})$. Along with the fact that $u''<0$ and (7d), the proof is complete. □

Proof of Corollary 1:

When there are only verifiable education expenses, we have $\frac{\partial y_1(\theta)}{\partial\theta}=0$. Then, equation (6a) implies that the standard (inverse) Euler equation holds. □

Proof of Proposition 3:

For the scenario when there are non-verifiable education expenses, then according to Lemma 1 and Proposition 1, $\lambda_2 > 0,\ \mu(\theta) < 0$ and $\frac{\partial y_1(\theta)}{\partial\theta} < 0$. Along with the fact that $u'' < 0$ and $u' > 0$, equation (6b) implies

$$\tau_{z_1}(\theta) \equiv 1 - \frac{\phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{w_1 h_1}}{u'\left(c_1(\theta)\right)} = \frac{\beta\mu(\theta)u'\left(c_2(\theta)\right)u''\left(c_1(\theta)\right)}{\lambda_2\pi(\theta)u'\left(c_1(\theta)\right)}\frac{\partial y_1(\theta)}{\partial\theta} < 0 \tag{22a}$$

for any $\theta \in (\underline{\theta}, \overline{\theta})$. Moreover, based on equation (6c), the facts $\frac{\partial h_2(\theta)}{\partial\theta} > 0,\ \phi' > 0$ and $\phi'' > 0$ imply

$$\tau_{z_2}(\theta) = 1 - \frac{\phi'\left(\frac{z_2(\theta)}{h_2}\right)\frac{1}{w_2 h_2}}{u'\left(c_2(\theta)\right)} = \frac{-\beta\mu(\theta)}{\lambda_2\pi(\theta)\left[h_2(\theta)\right]^2 w_2}\left[\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + \phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\frac{\partial h_2(\theta)}{\partial\theta} > 0 \tag{22b}$$

for any $\theta \in (\underline{\theta}, \overline{\theta})$. □

Proof of Corollary 2:

When there are only verifiable education expenses, the result $\frac{\partial y_1(\theta)}{\partial\theta} = 0$ implies $\tau_{z_1}(\theta) = 0$ in equation (22a) and $\tau_{z_2}(\theta) > 0$ in equation (22b). □

Proof of Proposition 4:

Adding $\frac{-\beta\psi_x(x_1(\theta),y_1(\theta))z_2(\theta)}{[h_2(\theta)]^2}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)$ on the both sides of (6d), then equation (6d) gives

$$\begin{aligned} &u'\left(c_1(\theta)\right) - \frac{\beta\psi_x\left(x_1(\theta),y_1(\theta)\right)z_2(\theta)}{\left[h_2(\theta)\right]^2}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right) \\ &= \frac{\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\left[h_2(\theta)\right]^2}\left[\frac{u'\left(c_1(\theta)\right)}{\beta R_2 u'\left(c_2(\theta)\right)} - 1\right] - \frac{\beta\mu(\theta)u'\left(c_1(\theta)\right)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\lambda_1\pi(\theta)\left[h_2(\theta)\right]^3}\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + 2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}. \end{aligned} \tag{23a}$$

Dividing $u'(c_1(\theta))$ on the both sides of (23a) and based on definition of (7c), then equation (23a) becomes

$$\begin{aligned} \tau_{x_1}(\theta) &= 1 - \frac{\beta\psi_x\left(x_1(\theta),y_1(\theta)\right)\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)}{u'\left(c_1(\theta)\right)\left[h_2(\theta)\right]^2} \\ &= \frac{\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\left[h_2(\theta)\right]^2}\left[\frac{1}{\beta R_2 u'\left(c_2(\theta)\right)} - \frac{1}{u'\left(c_1(\theta)\right)}\right] - \frac{\beta\mu(\theta)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\lambda_1\pi(\theta)\left[h_2(\theta)\right]^3}\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + 2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta} \end{aligned}$$

Using equation (8a) to replace $\frac{1}{u'(c_1(\theta))} - \frac{1}{\beta R_2 u'(c_2(\theta))}$ by Ω_1, then

$$\tau_{x_1}(\theta) = \frac{-\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\left[h_2(\theta)\right]^2}\Omega_1 - \frac{\beta\mu(\theta)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\lambda_1\pi(\theta)\left[h_2(\theta)\right]^3}\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + 2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}$$

Using Proposition 3 and equation (20e), the above equation can be rewritten as

$$\tau_{x_1}(\theta) = \frac{-\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\left[h_2(\theta)\right]^2}\Omega_1 + \frac{z_2(\theta)w_2\psi_x\left(x_1(\theta),y_1(\theta)\right)}{R_2 h_2(\theta)}\tau_{z_2}(\theta) - \frac{\beta\mu(\theta)z_2(\theta)\psi_x\left(x_1(\theta),y_1(\theta)\right)}{\lambda_1\pi(\theta)\left[h_2(\theta)\right]^3}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{\partial h_2(\theta)}{\partial\theta} \tag{23b}$$

Using the notations in Definition 2, the first term of (23b) can be replaced by $-\mathcal{K}_2(\theta)$, and the second term of (23b) can be replaced by $\mathcal{N}_2(\theta)$. Therefore, the equation (23b) can be rewritten as

$$\tau_{x_1}(\theta) = -\mathcal{K}_2(\theta) + \mathcal{N}_2(\theta) - \frac{\beta\mu(\theta)z_2(\theta)\psi_x(x_1(\theta),y_1(\theta))}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{\partial h_2(\theta)}{\partial\theta}.$$

Hence, according to Definition 2, the above equation implies that the net human capital wedge is as follows:

$$\tau^n_{x_1}(\theta) \equiv \tau_{x_1}(\theta) - \mathcal{N}_2(\theta) + \mathcal{K}_2(\theta) = \frac{-\beta\mu(\theta)z_2(\theta)\psi_x(x_1(\theta),y_1(\theta))}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{\partial h_2(\theta)}{\partial\theta} > 0$$

Due to the fact that $\mu(\theta) < 0$ *for* $\theta \in (\underline{\theta}, \overline{\theta})$ (Lemma 1), the net human capital wedge is positive for any $\theta \in (\underline{\theta}, \overline{\theta})$. □

A.5 Proof of Proposition 5 in Subsection 4.1

Suppose that an agent with the type history $\theta^t = (\theta_1, \theta_2, \ldots, \theta_t)$ reports type $\sigma = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_t)$. The agent chooses non-verifiable education expenses $y^\sigma(\theta^t)$ to maximize the following problem:

$$W^\sigma(\theta^t) = \max_y u\left(c(\tilde{\theta}^t) + y(\tilde{\theta}^t) - y\right) - \phi\left(\frac{z(\tilde{\theta}^t)}{h(\theta^{t-1})}\right)$$
$$+\beta\int\left[u\left(c(\tilde{\theta}^{t+1})\right) - \phi\left(\frac{z(\tilde{\theta}^{t+1})}{\psi\left(x(\tilde{\theta}^t), y\right) + \theta_t}\right) + \beta\int W^\sigma(\theta^{t+2})\pi(\theta_{t+2})d\theta_{t+2}\right]\pi(\theta_{t+1})d\theta.$$

The first-order condition with respect to y is

$$-u'\left(c(\tilde{\theta}^t) + y(\tilde{\theta}^t) - y^\sigma(\theta^t)\right) - \beta\int\phi_h\left(\frac{z(\tilde{\theta}^{t+1})}{h^\sigma(\theta^t)}\right)\psi_y\left(x(\tilde{\theta}^t), y^\sigma(\theta^t)\right)\pi(\theta_{t+1})d\theta_{t+1} = 0,$$

where $c^\sigma(\theta^t) + y^\sigma(\theta^t) = c(\tilde{\theta}^t) + y(\tilde{\theta}^t)$ and $\phi_h\left(\frac{z(\tilde{\theta}^{t+1})}{h^\sigma(\theta^t)}\right) = -\phi'\left(\frac{z(\tilde{\theta}^{t+1})}{h^\sigma(\theta^t)}\right)\frac{z(\tilde{\theta}^{t+1})}{[h^\sigma(\theta^t)]^2} < 0$. □

A.6 Extension of a one-deviation strategy to multi-deviation strategies

The Appendix shows that, under proper conditions, that the incentive compatibility of the one-deviation strategy can be extended to a multi-deviation strategy. The goal is to show that if a one-deviation strategy cannot generate a higher lifetime utility than a truth-telling strategy, then neither can multi-deviation strategies. In other words, given the true type θ^T, suppose that the best strategy among all one-deviation strategies $\hat{\sigma} = (\theta_1, \ldots, \hat{\theta}_t, \ldots, \theta_T)$ is to tell the truth; namely,

$$W(\theta^t) = \max_{\hat{\theta}} W^{\hat{\sigma}}(\theta^t).$$

Then, telling the truth is also the best strategy among all multi-deviation strategies.

Without loss of generality, suppose a strategy σ that reports the type truthfully in all periods except in $t = t_1, \ldots, t_m$. To be more specific, denote the reporting type under strategy σ as σ^T, which follows the following rule.

$$\sigma^T = \{(\sigma_1, \ldots, \sigma_T) \mid \sigma_s = \theta_s, \forall s \notin \{t_1, \ldots, t_m\}\}.$$

We will show that, even allowing for the possibility of deviating in the future periods, the agent will choose to tell the truth; that is,

$$W\left(\theta^{t_1}\right)=\max_{\sigma_{t_1},\ldots,\sigma_{t_m}} W^{\sigma}\left(\theta^{t_1}\right).$$

Before we proceed to the proof, we establish a result in the following Lemma.

Lemma 2. $y^{\sigma}(\theta^t)=y^{\sigma}\left(\tilde{\theta}^{t-1},\theta_t\right)$ *and* $c^{\sigma}(\theta^t)=c^{\sigma}\left(\tilde{\theta}^{t-1},\theta_t\right)$ *for any* $\tilde{\theta}^{t-1}\in\Theta^{t-1}$ *and any given strategy* σ.

Proof: According to Proposition 5, the optimal condition to determine the value of $y=y^{\sigma}(\theta^t)$ is

$$u'\left(c\left(\sigma^t\right)+y\left(\sigma^t\right)-y\right)=-\beta\int\phi_h\left(\frac{z\left(\sigma^{t+1}\right)}{\psi\left(x\left(\sigma^t\right),y\right)+\theta_t}\right)\left[\psi_y\left(x\left(\sigma^t\right),y\right)\right]\pi(\theta_{t+1})d\theta_{t+1}.$$

As can be seen, this condition does not depend on the past history of true types θ^{t-1}, and hence the optimal condition for $y=y^{\sigma}\left(\tilde{\theta}^{t-1},\theta_t\right)$ is the same as the above equation. Therefore, $y^{\sigma}(\theta^t)=y^{\sigma}\left(\tilde{\theta}^{t-1},\theta_t\right)$. Based on the fact that $c^{\sigma}(\theta^t)=c(\sigma^t)+y(\sigma^t)-y^{\sigma}(\theta^t)$, the property $y^{\sigma}(\theta^t)=y^{\sigma}\left(\tilde{\theta}^{t-1},\theta_t\right)$ implies $c^{\sigma}(\theta^t)=c^{\sigma}\left(\tilde{\theta}^{t-1},\theta_t\right)$. □

Assumption 2. Suppose that one-deviation strategies satisfy the following conditions.

(i) Among all one-deviation strategies $\hat{\sigma}=\left(\theta_1,\ldots,\hat{\theta}_t,\ldots,\theta_T\right)$, telling the truth is the best reporting strategy:

$$W(\theta^t)\geq W^{\hat{\sigma}}(\theta^t).$$

(ii) Given any current human capital level h_t, the lifetime utility $\widetilde{W}^{\hat{\sigma}}(\theta^t;h_t)$ is defined as

$$\widetilde{W}^{\hat{\sigma}}(\theta^t;h_t)=u\left(c^{\hat{\sigma}}(\theta^t)\right)-\phi\left(\frac{z(\theta^{t-1},\hat{\theta}_t)}{h_t}\right)+\beta\int W^{\hat{\sigma}}(\theta^{t+1})\pi(\theta_{t+1})d\theta_{t+1}.$$

Condition (i) still applies to $\widetilde{W}^{\hat{\sigma}}(\theta^t;h_t)$; that is, $\widetilde{W}(\theta^t;h_t)\geq\widetilde{W}^{\hat{\sigma}}(\theta^t;h_t)$.

The incentive compatibility condition in the text assures that Condition (i) is met. Moreover, Condition (ii) assumes that, just like a predetermined human capital in the first period, when the human capital level is given in period t, then a one-deviation strategy cannot generate a higher utility than a truth-telling strategy.

We are ready to prove that if a one-deviation strategy cannot lead to a higher utility than a truth-telling strategy, then neither can a multi-deviation strategy.

Proposition 11. *Under Assumption 2, for any multi-deviation strategy σ, which reports the type truthfully in all periods except $t=t_1,\ldots,t_m$, where $t_m>\cdots>t_1$, the best reporting type in the first possible-deviation period $t=t_1$ is to tell the truth. That is, $\sigma^*_{t_1}=\theta_{t_1}$ and thus, $W(\theta^{t_1})=\max_{\sigma_{t_1},\ldots,\sigma_{t_m}} W^{\sigma}(\theta^{t_1})$.*

Proof: We start from the last possible deviation period t_m. First, we show that the best strategy in period t_m is to report the true type; that is, $\sigma^*_{t_m}=\theta_{t_m}$.

Given a human capital level h_t, the lifetime utility of an agent with type θ^{t_m} in period t_m is

$$\tilde{W}^{\sigma}\left(\theta^{t_m};h_{t_m}\right)=u\left(c^{\sigma}\left(\theta^{t_m}\right)\right)-\phi\left(\frac{z\left(\sigma^{t_m}\right)}{h_{t_m}}\right)+\beta\int W^{\sigma}\left(\theta^{t_m+1}\right)\pi\left(\theta_{t_m+1}\right)d\theta_{t_m+1}$$

$$=u\left(c^{\sigma}\left(\theta^{t_m}\right)\right)-\phi\left(\frac{z\left(\sigma^{t_m}\right)}{h_{t_m}}\right)$$

$$+\beta^{s-t_m}\int\sum_{s=t_m+1}^{T}\left[u\left(c^{\sigma}\left(\theta^{s}\right)\right)-\phi\left(\frac{z\left(\sigma^{s}\right)}{\psi\left(x\left(\sigma^{s-1}\right),y^{\sigma}\left(\theta^{s-1}\right)\right)+\theta_{s-1}}\right)\right]\pi\left(\theta_{t_m+1}\right)\ldots\pi\left(\theta_{s}\right)d\theta_{t_{m+1}}\ldots d\theta_{s}.$$

The lifetime utility of another agent with type $(\sigma^{t_m-1},\theta_{t_m})$ in period t_m is

$$\tilde{W}^{\sigma}\left((\sigma^{t_m-1},\theta_{t_m});h_{t_m}\right)=u\left(c^{\sigma}(\sigma^{t_m-1},\theta_{t_m})\right)-\phi\left(\frac{z(\sigma^{t_m})}{h_{t_m}}\right)+\beta\int W^{\sigma}(\sigma^{t_m-1},\theta_{t_m},\theta_{t_m+1})\pi(\theta_{t_m+1})d\theta_{t_m+1}$$

$$=u\left(c^{\sigma}(\sigma^{t_m-1},\theta_{t_m})\right)-\phi\left(\frac{z(\sigma^{t_m})}{h_{t_m}}\right)$$

$$+\beta^{s-t_m}\int\sum_{s=t_m+1}^{T}\left[u\left(c^{\sigma}(\sigma^{t_m-1},\theta_{t_m},\ldots\theta_{s})\right)-\phi\left(\frac{z(\sigma^{s})}{\psi\left(x(\sigma^{s-1}),y^{\sigma}(\sigma^{t_m-1},\theta_{t_m},\ldots\theta_{s-1})\right)+\theta_{s-1}}\right)\right]\pi(\theta_{t_m+1})\ldots\pi(\theta_{s})d\theta_{t_{m+1}}\ldots d\theta_{s}.$$

By Lemma 2, $y^{\sigma}(\theta^{s})=y^{\sigma}(\sigma^{s-1},\theta_s)$ and $c^{\sigma}(\theta^{s})=c^{\sigma}(\sigma^{s-1},\theta_s)$. Then, from the above equations, one can easily show that, given the same reporting strategy σ in which t_m is the last possible deviation period, the agent with type θ^{t_m} and the other agent with type $(\sigma^{t_m-1},\theta_{t_m})$ have the same the lifetime utility; that is,

$$\tilde{W}^{\sigma}\left(\theta^{t_m};h_{t_m}\right)=\tilde{W}^{\sigma}\left((\sigma^{t_m-1},\theta_{t_m});h_{t_m}\right).$$

Note that, for the agent with type $(\sigma^{t_m-1},\theta_{t_m})$, his reporting strategy σ can be seen as a one-deviation strategy, which reports the type truthfully for all periods except t_m. Then, Condition (ii) in Assumption 2 applies; that is,

$$\theta_{t_m}\in\arg\max_{\sigma_{t_m}}\tilde{W}^{\sigma}\left((\sigma^{t_m-1},\theta_{t_m});h_{t_m}\right).$$

Since $\tilde{W}^{\sigma}(\theta^{t_m};h_{t_m})=\tilde{W}^{\sigma}((\sigma^{t_m-1},\theta_{t_m});h_{t_m})$, we also obtain

$$\theta_{t_m}\in\arg\max_{\sigma_{t_m}}\tilde{W}^{\sigma}\left(\theta^{t_m};h_{t_m}\right).$$

This completes the proof that, in the last possible deviation period t_m, the best reporting strategy is to tell the truth. Thus, no matter whether an agent reports his types truthfully or not in periods before t_m, the agent obtains the same utility if the type is reported truthfully in period t_m. This is true for any given level of human capital h_{t_m} in period t_m. Although agents with different reporting strategies may generate different values of h_{t_m}, it would not change the fact that, in period t_m, the best reporting strategy is to tell the truth. Even if deviating in t_m may generate a higher utility from the perspectives in earlier periods, it is not a time consistent strategy, because it cannot be carried out in period t_m.

Next, following the same method, we can easily prove in periods by periods that, for all periods before t_m except t_1 (i.e., $t = t_{m-1}, \ldots, t_2$), the best reporting strategy is to tell the truth.

Finally, given that the agent will tell the truth in all future possible deviation periods $t = t_{m-1}, \ldots, t_2$, then in period t_1, the reporting strategy σ is a one-deviation strategy and thus, using Condition (i) in Assumption 2, the best strategy is to tell the truth. This completes the proof. □

A.7 Proof of Proposition 6 in Subsection 4.2

First, to prove part (1) of Proposition 6, Proposition 5 indicates that the optimal choice of the agent with reporting strategy $\hat{\sigma}$ is

$$u'\left(c\left(\hat{\theta}^t\right)+y\left(\hat{\theta}^t\right)-y^{\hat{\sigma}}\left(\theta^t\right)\right)=-\beta\int\phi_h\left(\frac{z\left(\hat{\theta}^{t+1}\right)}{h^{\hat{\sigma}}\left(\theta^t\right)}\right)\psi_y\left(x\left(\hat{\theta}^t\right),y^{\hat{\sigma}}\left(\theta^t\right)\right)\pi\left(\theta_{t+1}\right)d\theta_{t+1}, \tag{24a}$$

Differentiating equation (24a) with respect to θ_t yields

$$u''\left(c\left(\hat{\theta}^t\right)+y\left(\hat{\theta}^t\right)-y^{\hat{\sigma}}\left(\theta^t\right)\right)\frac{\partial y^{\hat{\sigma}}\left(\theta^t\right)}{\partial\theta_t}$$
$$=\beta\int\left[\phi_{hh}\left(\frac{z\left(\hat{\theta}^{t+1}\right)}{\psi\left(x\left(\hat{\theta}^t\right),y^{\hat{\sigma}}\left(\theta^t\right)\right)+\theta_t}\right)\psi_y^{\hat{\sigma}}\left(\theta^t\right)\left[1+\psi_y^{\hat{\sigma}}\left(\theta^t\right)\frac{\partial y^{\hat{\sigma}}\left(\theta^t\right)}{\partial\theta_t}\right]+\phi_h\left(\frac{z\left(\hat{\theta}^{t+1}\right)}{\psi\left(x\left(\hat{\theta}^t\right),y^{\hat{\sigma}}\left(\theta^t\right)\right)+\theta_t}\right)\psi_{yy}^{\hat{\sigma}}\left(\theta^t\right)\frac{\partial y^{\hat{\sigma}}\left(\theta^t\right)}{\partial\theta_t}\right]\pi\left(\theta_{t+1}\right)d\theta_{t+1},$$

where $\phi_{hh}\left(\frac{z(\hat{\theta}^{t+1})}{h^{\hat{\sigma}}(\theta^t)}\right)=\phi''\left(\frac{z(\hat{\theta}^{t+1})}{h^{\hat{\sigma}}(\theta^t)}\right)\frac{\left[z(\hat{\theta}^{t+1})\right]^2}{\left[h^{\hat{\sigma}}(\theta^t)\right]^4}+2\phi'\left(\frac{z(\hat{\theta}^{t+1})}{h^{\hat{\sigma}}(\theta^t)}\right)\frac{z(\hat{\theta}^{t+1})}{\left[h^{\hat{\sigma}}(\theta^t)\right]^3}>0$, $\psi_y^{\hat{\sigma}}(\theta^t)=\psi_y\left(x(\hat{\theta}^t),y^{\hat{\sigma}}(\theta^t)\right)>0$ and $\psi_{yy}^{\hat{\sigma}}(\theta^t)=\psi_{yy}\left(x\left(\hat{\theta}^t\right),y^{\hat{\sigma}}(\theta^t)\right)<0$.

Manipulation of this above condition gives

$$\frac{\partial y^{\hat{\sigma}}\left(\theta^t\right)}{\partial\theta_t}=\frac{\beta\int\left[\phi_{hh}\left(\frac{z\left(\hat{\theta}^{t+1}\right)}{\psi\left(x\left(\hat{\theta}^t\right),y^{\hat{\sigma}}\left(\theta^t\right)\right)+\theta_t}\right)\psi_y^{\hat{\sigma}}\left(\theta^t\right)\right]\pi\left(\theta_{t+1}\right)d\theta_{t+1}}{u''\left(c^{\hat{\sigma}}\left(\theta^t\right)\right)-\beta\int\left[\phi_{hh}\left(\frac{z\left(\hat{\theta}^{t+1}\right)}{h^{\hat{\sigma}}\left(\theta^t\right)}\right)\left[\psi_y^{\hat{\sigma}}\left(\theta^t\right)\right]^2+\phi_h\left(\frac{z\left(\hat{\theta}^{t+1}\right)}{h^{\hat{\sigma}}\left(\theta^t\right)}\right)\psi_{yy}^{\hat{\sigma}}\left(\theta^t\right)\right]\pi\left(\theta_{t+1}\right)d\theta_{t+1}}. \tag{24b}$$

Since $h^{\hat{\sigma}}(\theta^t)=\psi\left(x(\hat{\theta}^t),y^{\hat{\sigma}}(\theta^t)\right)+\theta_t$ and $\psi_{yy}<0<\psi_y$, along with the fact that $\phi_h<0$ and $\phi_{hh}>0$, the above equation implies that $\frac{\partial y^{\hat{\sigma}}(\theta^t)}{\partial\theta_t}<0$.

Moreover, using (24b), we obtain

$$\frac{dh^{\hat{\sigma}}(\theta^t)}{d\theta_t}=\psi_y^{\hat{\sigma}}(\theta^t)\frac{\partial y^{\hat{\sigma}}(\theta^t)}{\partial\theta_t}+1=\frac{u''\left(c^{\hat{\sigma}}(\theta^t)\right)-\beta\int\phi_h\left(\frac{z(\hat{\theta}^{t+1})}{h^{\hat{\sigma}}(\theta^t)}\right)\psi_{yy}^{\hat{\sigma}}(\theta^t)\pi(\theta_{t+1})d\theta_{t+1}}{u''\left(c^{\hat{\sigma}}(\theta^t)\right)-\beta\int\left[\phi_{hh}\left(\frac{z(\hat{\theta}^{t+1})}{h^{\hat{\sigma}}(\theta^t)}\right)\left[\psi_y^{\hat{\sigma}}(\theta^t)\right]^2+\phi_h\left(\frac{z(\hat{\theta}^{t+1})}{h^{\hat{\sigma}}(\theta^t)}\right)\psi_{yy}^{\hat{\sigma}}(\theta^t)\right]\pi(\theta^t)d\theta_{t+1}}>0$$

Next, to prove part (2) of Proposition 6, according to Proposition 5, for periods $s\geq t+1$, we find that, given the same reporting history $(\hat{\theta}_1,\ldots,\hat{\theta}_s)$, the agent with reporting strategy $\hat{\sigma}(\theta^s)$ and the truth-telling agent with type $\hat{\theta}^s$ share the same optimal condition to solve the optimal non-verifiable human capital investment y as follows.

$$u'\left(c\left(\hat{\theta}^s\right)+y\left(\hat{\theta}^s\right)-y\right)=-\beta\int\phi_h\left(\frac{z\left(\hat{\theta}^{s+1}\right)}{\psi\left(x\left(\hat{\theta}^s\right),y\right)+\theta_s}\right)\psi_y\left(x\left(\hat{\theta}^s\right),y\right)\pi\left(\theta_{s+1}\right)d\theta_{s+1}.$$

This implies that they have the same amount of non-verifiable human capital investment, and thus the same amount of consumption. That is, $y^{\hat{\sigma}}(\theta^s)=y(\hat{\theta}^s)$ and $c^{\hat{\sigma}}(\theta^s)=c(\hat{\theta}^s)$ for any $s\geq t+1$.□

A.8 The relaxed social planning problem in the subsection 4.3, derivation of the relaxed social planning problem and proofs of Propositions 7-9 and Corollaries 3-4 in Subsection 4.3

The relaxed social planning problem is set up as follows. First, for periods $t=2,3,\dots,T-1$, the expected resource-cost minimization problem is:

$$\mathcal{K}\left(v,\Delta,h,\theta_{\_},t\right)=\min\int_{\underline{\theta}}^{\overline{\theta}}\left[c(\theta)+y(\theta)+x(\theta)-w_t z(\theta)+\tfrac{1}{R_{t+1}}\mathcal{K}\left(v(\theta),\Delta(\theta),h(\theta),\theta,t+1\right)\right]\pi(\theta)d\theta, \quad (25a)$$

subject to

$$W(\theta)=u\left(c(\theta)\right)-\phi\left(\tfrac{z(\theta)}{h(\theta_{\_})}\right)+\beta v(\theta),$$

$$\dot{W}(\theta)=-u'\left(c(\theta)\right)\tfrac{\partial y(\theta)}{\partial\theta}+\beta\Delta(\theta),$$

where $v=\int W(\theta)\pi(\theta)\,d\theta$ and $\Delta=\int\phi'\left(\frac{z(\theta)}{h(\theta_{\_})}\right)\frac{z(\theta)\frac{\partial h(\theta_{\_})}{\partial\theta_{\_}}}{[h(\theta_{\_})]^2}\pi(\theta)\,d\theta$, with $\theta_{\_}$ denoting past shocks and the minimization being taken over $c(\theta),x(\theta),z(\theta),W(\theta),v(\theta)$ and $\Delta(\theta)$.[45]

For period 1, the problem is indexed by the targeted lifetime utility $\underline{v}$. The problem in $t=1$ is reformulated as follows.

$$\mathcal{K}\left(\underline{v},h,1\right)=\min\int\left[c(\theta)+x(\theta)+y(\theta)-w_1 z(\theta)+\tfrac{1}{R_2}\mathcal{K}\left(v(\theta),\Delta(\theta),h(\theta),\theta,2\right)\right]\pi(\theta)d\theta, \quad (25b)$$

subject to

$$W(\theta)=u\left(c(\theta)\right)-\phi\left(\tfrac{z(\theta)}{h_1}\right)+\beta v(\theta),$$

$$\dot{W}(\theta)=-u'\left(c(\theta)\right)\tfrac{\partial y(\theta)}{\partial\theta}+\beta\Delta(\theta),$$

$$\int W(\theta)\pi(\theta)d\theta\ \geq\ \underline{v},$$

where the minimization is taken over $c(\theta),x(\theta),z(\theta),W(\theta),v(\theta)$ and $\Delta(\theta)$.

We must note that the recursive formulation above has used the agent's envelope condition (11c), that involves the optimal non-verifiable human capital investment. As defined in (10b) concerning the expected lifetime utility under a reporting strategy $\sigma\in\mathcal{R}$, the optimal non-verifiable human capital investment $y(\theta)$ is chosen to satisfy the optimal condition in Proposition 5. Moreover, in solving the

[45] As mentioned in Section 3, we analytically solve this relaxed planning problem and then will numerically verify that the solution satisfies the incentive compatibility condition later.

relaxed social planning problem below, the properties of the optimal non-verifiable human capital investment characterized in Proposition 6 will be used.

We are ready to derive the relaxed social planning problem. Since the utility is given by $W(\theta) = u\big(c(\theta)\big) - \phi\left(\frac{z(\theta)}{h(\theta_-)}\right) + \beta v(\theta)$, we can replace $c(\theta)$ by $u^{-1}\left[W(\theta) - \beta v(\theta) + \phi\left(\frac{z(\theta)}{h(\theta_-)}\right)\right]$. Then, for periods $t = 2, 3, \ldots, T$, the Hamiltonian is as follows.

$$\begin{aligned}
\mathcal{K}\big(v(\theta_-),\Delta(\theta_-),h(\theta_-),\theta_-,t\big) = {} & \left\{u^{-1}[W(\theta)-\beta v(\theta)+\phi\left(\tfrac{z(\theta)}{h(\theta_-)}\right)] + x(\theta) + y(\theta) - w_t z(\theta) + \tfrac{\mathcal{K}(v(\theta),\Delta(\theta),h(\theta),\theta,t+1)}{R_{t+1}}\right\}\pi(\theta) \\
& + \lambda(\theta_-)\big[v(\theta_-) - W(\theta)\pi(\theta)\big] \\
& + \gamma(\theta_-)\left[\Delta(\theta_-) - \frac{z(\theta)\frac{\partial h(\theta_-)}{\partial \theta_-}}{\left[h(\theta_-)\right]^2}\phi'\left(\tfrac{z(\theta)}{h(\theta_-)}\right)\pi(\theta)\right] \\
& + \hat{\mu}(\theta)\left[-u'\left(u^{-1}\left[W(\theta)-\beta v(\theta)+\phi\left(\tfrac{z(\theta)}{h(\theta_-)}\right)\right]\right)\frac{\partial y(\theta)}{\partial \theta} + \beta\Delta(\theta)\right].
\end{aligned}$$

where $\lambda(\theta_-)$ and $\gamma(\theta_-)$ are the shadow price associated with $v(\theta_-)$ and $\Delta(\theta_-)$ respectively, and $\hat{\mu}(\theta)$ is the co-state variable associated with $\dot{W}(\theta)$.

The first-order conditions are

$$\frac{\partial \mathcal{K}}{\partial z(\theta)} = \pi(\theta)\left[\frac{\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)\frac{1}{h(\theta_-)}\left[1-\frac{\hat{\mu}(\theta)}{\pi(\theta)}u''\big(c(\theta)\big)\frac{\partial y(\theta)}{\partial \theta}\right]}{u'\big(c(\theta)\big)} - w_t - \frac{\gamma(\theta_-)\frac{\partial h(\theta_-)}{\partial \theta_-}\left[\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)+\frac{z(\theta)}{h(\theta_-)}\phi''\left(\frac{z(\theta)}{h(\theta_-)}\right)\right]}{\left[h(\theta_-)\right]^2}\right] = 0, \quad \text{(26a)}$$

$$\frac{\partial \mathcal{K}}{\partial x(\theta)} = \pi(\theta)\left[1 + \frac{\mathcal{K}_h\big(v(\theta),\Delta(\theta),h(\theta),\theta,t+1\big)\psi_x}{R_{t+1}}\right] = 0, \quad \text{(26b)}$$

$$\frac{\partial \mathcal{K}}{\partial \Delta(\theta)} = \frac{\pi(\theta)}{R_{t+1}}K_\Delta\big(v(\theta),\Delta(\theta),h(\theta),\theta_t,t+1\big) + \beta\hat{\mu}(\theta) = 0, \quad \text{(26c)}$$

$$\frac{\partial \mathcal{K}}{\partial v(\theta)} = \frac{-\beta\pi(\theta)}{u'\big(c(\theta)\big)} + \frac{\pi(\theta)}{R_{t+1}}K_v\big(v(\theta),\Delta(\theta),h(\theta),\theta_t,t+1\big) + \frac{\hat{\mu}(\theta)\beta u''\big(c(\theta)\big)}{u'\big(c(\theta)\big)}\frac{\partial y(\theta)}{\partial \theta} = 0. \quad \text{(26d)}$$

Moreover, the law of motion for the co-state $\mu(\theta)$ is:

$$\frac{\partial \mathcal{K}}{\partial W(\theta)} = -\dot{\hat{\mu}}(\theta) \equiv -\frac{\partial \hat{\mu}(\theta)}{\partial \theta} = \left[\frac{1}{u'\big(c(\theta)\big)} - \lambda(\theta_-)\right]\pi(\theta) - \frac{\hat{\mu}(\theta)u''\big(c(\theta)\big)}{u'\big(c(\theta)\big)}\frac{\partial y(\theta)}{\partial \theta}, \quad \text{(26e)}$$

with boundary conditions $\hat{\mu}(\bar{\theta}) = \hat{\mu}(\underline{\theta}) = 0$.

Envelope conditions are as follows.

$$\mathcal{K}_\Delta\big(v(\theta_-),\Delta(\theta_-),h(\theta_-),\theta_-,t\big) = \gamma(\theta_-), \quad \text{(26f)}$$

$$\mathcal{K}_v\big(v(\theta_-),\Delta(\theta_-),h(\theta_-),\theta_-,t\big) = \lambda(\theta_-), \quad \text{(26g)}$$

$$\mathcal{K}_h\big(v(\theta_-),\Delta(\theta_-),h(\theta_-),\theta_-,t\big) = \frac{z(\theta)\pi(\theta)}{\left[h(\theta_-)\right]^2}\left[\frac{-\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)\left[1-\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)}\frac{\partial y(\theta)}{\partial \theta}\right]}{u'(c(\theta))} + \frac{\gamma(\theta_-)\frac{\partial h(\theta_-)}{\partial \theta_-}\left[2\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)+\phi''\left(\frac{z(\theta)}{h(\theta_-)}\right)\frac{z(\theta)}{h(\theta_-)}\right]}{h(\theta_-)}\right]. \quad \text{(26h)}$$

Lagging conditions (26f), (26g) and (26h) by one period, we use (26b), (26c) and (26d) to obtain

$$\gamma(\theta)=\frac{-R_{t+1}\beta\hat{\mu}(\theta)}{\pi(\theta)}, \tag{26i}$$

$$\lambda(\theta)=\beta R_{t+1}\left[\frac{1}{u'(c(\theta))}-\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial\theta}\right], \tag{26j}$$

$$\frac{R_{t+1}}{\psi_x(\theta)}=\frac{z(\theta')\pi(\theta')}{\left[h(\theta)\right]^2}\left\{\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\left[\frac{1}{u'(c(\theta'))}-\frac{\hat{\mu}(\theta')u''(c(\theta'))}{\pi(\theta')u'(c(\theta'))}\frac{\partial y(\theta')}{\partial\theta'}\right]-\frac{\gamma(\theta)\frac{\partial h(\theta)}{\partial\theta}\left[2\phi'\left(\frac{z(\theta')}{h(\theta)}\right)+\phi''\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{h(\theta)}\right]}{h(\theta)}\right\}. \tag{26k}$$

In order to determine the sign of the optimal wedge, following Stantcheva (2017) we make the following assumption.[46]

Assumption 3. $v(\theta)$ is increasing in θ and $\mathcal{K}(v,\Delta,h,\theta,t)$ is increasing and convex in v. That is, for all θ, $(\partial/\partial v)\mathcal{K}\geq 0$ and $(\partial^2/\partial v^2)\mathcal{K}\geq 0$.

According to Assumption 3, agents with a higher shock θ today tend to get the allocation yielding a higher the expected future utility $v(\theta)$. Agents with a higher expected future utility v will have a higher expected discounted cost of providing an allocation $\mathcal{K}$, and moreover, the cost is increasing in v.

For any $\theta_0\in[\underline{\theta},\overline{\theta}]$, let the corresponding expected future utility be denoted by $v(\theta_0)\equiv v_0$. Then, the inverse function of v implies that $\theta_0=v^{-1}(v_0)$. Based on the envelope condition (26g), we find that

$$\frac{\partial\mathcal{K}}{\partial v_0}=\lambda(\theta_0)=\lambda\left(v^{-1}(v_0)\right). \tag{27a}$$

Then, taking the derivative with respect to v_0 on both sides of the equation (27a) gives

$$\frac{\partial^2\mathcal{K}}{\partial v_0^2}=\lambda'\left(v^{-1}(v_0)\right)\cdot\frac{d}{dv_0}v^{-1}(v_0)=\lambda'(\theta_o)\frac{1}{v'(\theta_0)}. \tag{27b}$$

By Assumption 3, (27b) implies that $\lambda'(\theta_0)>0$ for any $\theta_0\in[\underline{\theta},\overline{\theta}]$.

Building on Assumption 3, we easily establish Lemma 3 in the following, which proves that the co-state $\hat{\mu}(\theta)$ is positive, indicating a marginal cost caused by informational frictions.[47]

Lemma 3. *Under Assumption 3*, $\gamma(\theta)<0<\hat{\mu}(\theta)$ *for any* $\theta\in(\underline{\theta},\overline{\theta})$

Proof of Lemma 3.

---

[46] See Assumption 3 in Stantcheva (2017, p. 1978).

[47] The informational friction tends to reduce the social welfare or to raise the social cost. That is the reason the co-states are negative in the utility maximization problem (c.f. Lemma 1) and are positive in the cost minimization problem (c.f. Lemma 3.)

From (26j) and the envelope condition (26g), we obtain

$$\frac{\partial \mathcal{K}}{\partial v}=\lambda(\theta)=\beta R_{t+1}\left[\frac{1}{u'(c(\theta))}-\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}\right].$$

By Assumption 3, we know that $\lambda(\theta)$ is increasing in θ, so the term $\frac{1}{u'(c(\theta))}-\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}$ increases in θ as well.

With boundary conditions $\hat{\mu}(\overline{\theta})=\hat{\mu}(\underline{\theta})=0$, according to Rolle's theorem, there exists $\theta^*\in(\underline{\theta},\overline{\theta})$ such that $\dot{\hat{\mu}}(\theta^*)=0$. Then, the law of motion (26e) gives

$$\lambda(\theta_-)=\frac{1}{u'(c(\theta^*))}-\frac{\hat{\mu}(\theta^*)u''(c(\theta^*))}{\pi(\theta^*)u'(c(\theta^*))}\frac{\partial y(\theta^*)}{\partial \theta^*}. \tag{28}$$

Because $\frac{1}{u'(c(\theta))}-\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}$ is monotone increasing in θ, (28) implies that

$$\frac{1}{u'(c(\theta))}-\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}<\lambda(\theta_-)\quad \forall\, \theta<\theta^*,$$

$$\frac{1}{u'(c(\theta))}-\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}>\lambda(\theta_-)\quad \forall\, \theta>\theta^*.$$

Integrating (26e) and using the boundary condition $\hat{\mu}(\overline{\theta})=0$ we obtain

$$\hat{\mu}(\theta)=\int_{\theta}^{\overline{\theta}}\left[\frac{1}{u'(c(\theta''))}-\lambda(\theta_-)\right]\pi(\theta'')-\frac{\hat{\mu}(\theta'')u''(c(\theta''))}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial \theta''}d\theta''.$$

Note that this equation integrates over non-negative variables only for $\theta\in[\theta',\overline{\theta})$. Thus, the above equation implies $\hat{\mu}(\theta)>0$ for $\theta\in[\theta',\overline{\theta})$.

Similarly, using the boundary condition $\hat{\mu}(\underline{\theta})=0$, we get

$$\hat{\mu}(\theta)=\int_{\underline{\theta}}^{\theta}\left[\frac{-1}{u'(c(\theta''))}+\lambda(\theta_-)\right]\pi(\theta'')+\frac{\hat{\mu}(\theta'')u''(c(\theta''))}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial \theta''}d\theta'',$$

which also implies $\hat{\mu}(\theta)>0$ for $\theta\in(\underline{\theta},\theta']$.
From (26i), $\hat{\mu}(\theta)>0$ implies $\gamma(\theta)<0$ for $\theta\in(\underline{\theta},\theta']$. □

Proofs of Proposition 7 and Corollary 3

Now, we prove Proposition 7 in Subsection 4.3. First, integrating (26e) and using the boundary condition $\hat{\mu}(\bar{\theta})=0$ yields:

$$\hat{\mu}(\theta)=\int_{\theta}^{\overline{\theta}}\left[\frac{1}{u'(c(\theta''))}-\lambda(\theta_-)\right]\pi(\theta'')-\frac{\hat{\mu}(\theta'')u''(c(\theta''))}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial \theta''}d\theta''. \tag{29a}$$

Next, using (26j) to replace $\lambda(\theta_-)$ in (29a), we get

$$\hat{\mu}(\theta)=\int_{\theta}^{\bar{\theta}}\frac{\pi(\theta'')}{u'(c(\theta''))}-\frac{\hat{\mu}(\theta'')u''(c(\theta''))}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial\theta''}d\theta''-\beta R_t\left[\frac{1}{u'(c(\theta_-))}-\frac{\hat{\mu}(\theta_-)u''(c(\theta_-))}{\pi(\theta_-)u'(c(\theta_-))}\frac{\partial y(\theta_-)}{\partial\theta_-}\right]. \quad (29b)$$

Moreover, using boundary condition $\hat{\mu}(\underline{\theta})=0,$ (29b) leads to

$$\frac{1}{u'(c(\theta_-))}-\frac{1}{\beta R_t}\int_{\underline{\theta}}^{\bar{\theta}}\frac{\pi(\theta'')}{u'(c(\theta''))}d\theta''=\underbrace{\frac{\hat{\mu}(\theta_-)u''(c(\theta_-))}{\pi(\theta_-)u'(c(\theta_-))}\frac{\partial y(\theta_-)}{\partial\theta_-}}_{Current\ period's\ HCI\ effect}-\underbrace{\frac{1}{\beta R_t}\int_{\underline{\theta}}^{\bar{\theta}}\frac{\hat{\mu}(\theta'')u''(c(\theta''))}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial\theta''}d\theta''}_{Next\ period's\ HCI\ effect} \quad (29c)$$

With some manipulation, the above equation gives an inverse Euler equation as follows.

$$\frac{1}{u'(c(\theta_-))}-\frac{1}{\beta R_t}E\left[\frac{1}{u'(c(\theta))}\right]=\frac{\hat{\mu}(\theta_-)u''(c(\theta_-))}{\pi(\theta_-)u'(c(\theta_-))}\frac{\partial y(\theta_-)}{\partial\theta_-}-\frac{1}{\beta R_t}E\left[\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial\theta}\right].$$

If there are only verifiable education expenses, $\frac{\partial y(\theta_-)}{\partial\theta_-}=\frac{\partial y(\theta)}{\partial\theta}=0$ and the right-hand side of (29c) is zero. Then, the above equation reduces to the inverse Euler equation and the capital wedge is positive that involves the effect of insurance purposes. This is complete the proof of Corollary 3. However, when there are non-verifiable education expenses, the right-hand side of (29c) is not zero and the inverse Euler equation does not hold. In this case, agents' consumption may be increased by reducing non-verifiable human capital investment, the *HCI effect* in the text. The right-hand of (29c) are the HCI effects in current and the next periods that offset each other. It is not clear whether or not the HCI effect in next period is sufficiently strong, so the net effect on the capital wedge is ambiguous. However, in the terminal period, there is only the current HCI effect, so (29c) becomes

$$\frac{1}{u'(c(\theta_-))}-\frac{1}{\beta R_T}E\left[\frac{1}{u'(c(\theta))}\right]=\frac{\hat{\mu}(\theta_-)u''(c(\theta_-))}{\pi(\theta_-)u'(c(\theta_-))}\frac{\partial y(\theta_-)}{\partial\theta_-}. \quad (29d)$$

By Lemma 3 and Proposition 6, we have $\hat{\mu}(\theta)>0$ and $\frac{\partial y(\theta)}{\partial\theta}<0$. Then, (29d) gives

$$\frac{1}{u'(c(\theta_-))}>\frac{1}{\beta R_T}E\left[\frac{1}{u'(c(\theta))}\right],$$

which implies a larger capital wedge than the case with no non-verifiable education expenses. This proves Proposition 7. □

Proofs of Proposition 8 and Corollary 4

Next, we prove Proposition 8 in Subsection 4.3. For the first period $t=1$, the Hamiltonian of the social planning problem is:

$$
\begin{aligned}
\mathcal{K}(\underline{v},h_1,1)=&\left\{u^{-1}\left[W(\theta)-\beta v(\theta)+\phi\left(\tfrac{z(\theta)}{h_1}\right)\right]+x(\theta)+y(\theta)-w_1 z(\theta)+\tfrac{1}{R_2}\mathcal{K}\left(v(\theta),\Delta(\theta),h(\theta),\theta,2\right)\right\}\pi(\theta)\\
&+\lambda\left[\underline{v}-W(\theta)\pi(\theta)\right]\\
&+\hat{\mu}(\theta)\left[-u'\left(u^{-1}\left[W(\theta)-\beta v(\theta)+\phi\left(\tfrac{z(\theta)}{h_1}\right)\right]\right)\frac{\partial y(\theta)}{\partial\theta}+\beta\Delta(\theta)\right].
\end{aligned}
$$

The first-order conditions are

$$
\frac{\partial\mathcal{K}}{\partial z(\theta)}=\left[\frac{\phi'\left(\frac{z(\theta)}{h_1}\right)\frac{1}{h_1}}{u'(c(\theta))}-w_1\right]\pi(\theta)-\hat{\mu}(\theta)\frac{u''(c(\theta))\phi'\left(\frac{z(\theta)}{h_1}\right)\frac{1}{h_1}}{u'(c(\theta))}\frac{\partial y(\theta)}{\partial\theta}=0\,.
$$

The above condition gives the following labor wedge in the first period.

$$
\tau_z(\theta)=1-\frac{\phi'\left(\frac{z(\theta)}{h_1}\right)\frac{1}{w_1 h_1}}{u'(c(\theta))}=-\frac{\hat{\mu}(\theta)}{\pi(\theta)}\frac{u''(c(\theta))\phi'\left(\frac{z(\theta)}{h_1}\right)\frac{1}{h_1}}{w_1 u'(c(\theta))}\frac{\partial y(\theta)}{\partial\theta}.
$$

Similar to Lemma 3, it is easy to prove that $\tilde{\mu}(\theta)>0$. Then, we obtain $\frac{\partial y(\theta)}{\partial\theta}=0$ if there are only verifiable education expenses. However, based on Proposition 6, we obtain $\frac{\partial y(\theta)}{\partial\theta}<0$ if there are non-verifiable education expenses. Thus, in the first period, the labor wedge is zero when there are only verifiable education expenses, but is negative when there are non-verifiable education expenses.

On the other hand, for periods $t=2,3,\ldots,T$, from (26a), it is easy to obtain the labor wedge as follows.

$$
\tau_{z_t}(\theta)=\underbrace{\frac{-\gamma(\theta_-)\frac{\partial h(\theta_-)}{\partial\theta_-}\left[\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)+\frac{z(\theta)}{h(\theta_-)}\phi''\left(\frac{z(\theta)}{h(\theta_-)}\right)\right]}{w_t\left[h(\theta_-)\right]^2}}_{shirk-preventing\ effect\ \geq 0}+\underbrace{\frac{-\hat{\mu}(\theta)u''(c(\theta))\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)\frac{\partial y(\theta)}{\partial\theta}}{\pi(\theta)u'(c(\theta))w_t h(\theta_-)}}_{skill-fostering\ effect\ \leq 0}.
$$

Then, if there are only verifiable education expenses, we obtain $\frac{\partial y(\theta)}{\partial\theta}=0$, which implies that the last term of the above equation is zero, and thus the labor wedge is positive, based on Lemma 3. However, if there are non-verifiable education expenses, we obtain $\frac{\partial y(\theta)}{\partial\theta}<0$ based on Proposition 6. Then, the sign of the labor wedge is ambiguous except in the terminal period. Moreover, in the terminal period, agents do not invest in human capital, and thus $\frac{\partial y(\theta)}{\partial\theta}=0$. Therefore, no matter whether there are non-verifiable education expenses, the labor wedge is unambiguously positive in the terminal period. □

Proof of Proposition 9

Divided $\frac{R_{t+1}}{\psi_x(\theta)}$ on the both sides of (26k), we obtain the following equation

$$
1=\frac{z(\theta')\pi(\theta')\psi_x(\theta)}{\left[h(\theta)\right]^2 R_{t+1}}\left\{\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\left[\frac{1}{u'(c(\theta'))}-\frac{\hat{\mu}(\theta')u''(c(\theta'))}{\pi(\theta')u'(c(\theta'))}\frac{\partial y(\theta')}{\partial\theta'}\right]-\frac{\gamma(\theta)\left[2\phi'\left(\frac{z(\theta')}{h(\theta)}\right)+\phi''\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{h(\theta)}\right]\frac{\partial h(\theta)}{\partial\theta}}{h(\theta)}\right\} \tag{30a}
$$

Adding $\frac{-\beta\psi_x(\theta)}{u'(c(\theta))}\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{[h(\theta)]^2}\pi(\theta')$ on the both sides of (30a), then we have

$$1-\frac{\beta\psi_x(\theta)}{u'(c(\theta))}\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')\pi(\theta')}{\left[h(\theta)\right]^2}=\frac{z(\theta')\pi(\theta')\psi_x(\theta)\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\left[\frac{1}{u'(c(\theta'))}-\frac{\beta R_{t+1}}{u'(c(\theta))}-\frac{\hat{\mu}(\theta')u''(c(\theta'))}{\pi(\theta')u'(c(\theta'))}\frac{\partial y(\theta')}{\partial\theta'}\right]}{\left[h(\theta)\right]^2R_{t+1}}-\frac{z(\theta')\pi(\theta')\psi_x(\theta)\gamma(\theta)\left[2\phi'\left(\frac{z(\theta')}{h(\theta)}\right)+\phi''\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{h(\theta)}\right]\frac{\partial h(\theta)}{\partial\theta}}{\left[h(\theta)\right]^3R_{t+1}} \tag{30b}$$

Integrating (30b) with respect to θ' and according to the definition (12c), we get

$$\begin{aligned}\tau_x(\theta)&\equiv1-\frac{\beta\psi_x(\theta)}{u'(c(\theta))}\int\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{\left[h(\theta)\right]^2}\pi(\theta')d\theta'\\&=\int\frac{z(\theta')\psi_x(\theta)\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\left[\frac{1}{u'(c(\theta'))}-\frac{\beta R_{t+1}}{u'(c(\theta))}-\frac{\hat{\mu}(\theta')u''(c(\theta'))}{\pi(\theta')u'(c(\theta'))}\frac{\partial y(\theta')}{\partial\theta'}\right]-z(\theta')\psi_x(\theta)\frac{\gamma(\theta)}{h(\theta)}\left[2\phi'\left(\frac{z(\theta')}{h(\theta)}\right)+\phi''\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{h(\theta)}\right]\frac{\partial h(\theta)}{\partial\theta}}{\left[h(\theta)\right]^2R_{t+1}}\pi(\theta')d\theta'\end{aligned} \tag{30c}$$

Replacing θ' and θ by θ^{t+1} and θ^t, respectively and using notation $\Omega_t=\Omega_t(c(\theta^{t-1}),c(\theta^t)$ defined in (7d), the equation (30c) becomes

$$\tau_x(\theta^t)=E_t\left[\frac{\psi_x(\theta^t)z(\theta^{t+1})}{\left[h(\theta^t)\right]^2R_{t+1}}\left[\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\left[-\beta R_{t+1}\Omega_t-\frac{\hat{\mu}(\theta^{t+1})u''(c(\theta^{t+1}))}{\pi(\theta_{t+1})u'(c(\theta^{t+1}))}\frac{\partial y(\theta^{t+1})}{\partial\theta_{t+1}}\right]-\frac{\gamma(\theta^t)\frac{\partial h(\theta^t)}{\partial\theta_t}}{h(\theta^t)}\left[2\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)+\phi''\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\frac{z(\theta^{t+1})}{h(\theta^t)}\right]\right]\right]$$

Using the equation (14) in Proposition 8, the above equation can be rewritten as follows

$$\tau_x(\theta^t)=E_t\left[\frac{-\psi_x(\theta^t)z(\theta^{t+1})}{\left[h(\theta^t)\right]^2}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\beta\Omega_t+\frac{\psi_x(\theta^t)z(\theta^{t+1})}{h(\theta^t)R_{t+1}}w_t\tau_z(\theta^{t+1})-\frac{\psi_x(\theta^t)z(\theta^{t+1})\gamma(\theta^t)}{\left[h(\theta^t)\right]^3R_{t+1}}\frac{\partial h(\theta^t)}{\partial\theta_t}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\right] \tag{30d}$$

Using the notations $\mathcal{N}_{t+1}=\frac{w_{t+1}z_{t+1}}{R_{t+1}h_{t+1}}\psi_x\tau_{z_{t+1}}$ and $\mathcal{K}_{t+1}=\frac{\beta z_{t+1}}{(h_{t+1})^2}\phi'\left(\frac{z_{t+1}}{h_{t+1}}\right)\psi_x\Omega_t$ that is defined in Definition 2, then the equation (30d) becomes

$$\tau_x(\theta^t)=E_t\left[-\mathcal{K}_{t+1}(\theta^{t+1})+\mathcal{N}_{t+1}(\theta^{t+1})-\frac{\psi_x(\theta^t)z(\theta^{t+1})\gamma(\theta^t)}{\left[h(\theta^t)\right]^3R_{t+1}}\frac{\partial h(\theta^t)}{\partial\theta_t}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\right] \tag{30e}$$

Based on the Definition 2 and equation (30e), the net human capital wedge is derived as follows

$$\tau_x^n(\theta^t)\equiv E_t\left[\tau_x(\theta^t)-\mathcal{N}_{t+1}(\theta^{t+1})+\mathcal{K}_{t+1}(\theta^{t+1})\right]=E_t\left[\frac{-\psi_x(x(\theta^t),y(\theta^t))z(\theta^{t+1})\gamma(\theta^t)}{\left[h(\theta^t)\right]^3R_{t+1}}\frac{\partial h(\theta^t)}{\partial\theta_t}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\right]$$

which completes the proof of Proposition 9. □

A.9 Proof of Proposition 10 in Subsection 5.3

In Subsection 5.3, the problem of the agent of type θ^t is to maximize the following life-time utility

$$\tilde{U}^t(\tilde{k}_t,\tilde{h}_t,\theta_t)=\max u(\tilde{c}_t)-\phi\left(\frac{\tilde{z}_t}{\tilde{h}_t}\right)+\beta E_t\left[\tilde{U}^{t+1}(\tilde{k}_{t+1},\tilde{h}_{t+1},\theta_{t+1})\right]$$

subject to

$$\tilde{c}_t+\tilde{x}_t+\tilde{y}_t+\tilde{k}_{t+1}\le w_t\tilde{z}_t+R_t\tilde{k}_t-\mathcal{T}_t,$$

$$\tilde{h}_{t+1}=\psi(\tilde{x}_t,\tilde{y}_t)+\theta_t,$$

where $E_t\left[\tilde{U}^{t+1}(\tilde{k}_{t+1},\tilde{h}_{t+1},\theta_{t+1})\right]=\int\tilde{U}^{t+1}(\tilde{k}_{t+1},\tilde{h}_{t+1},\theta_{t+1})\pi(\theta_{t+1})d\theta_{t+1}$, and the maximization is taken over $\{c_t,z_t,x_t,y_t,h_{t+1},k_{t+1}\}$.

If we restrict $x_t=x(\hat{\theta}^t)$ for some reporting strategy $\hat{\sigma}=(\hat{\theta}_1,\ldots,\hat{\theta}_t)\in\hat{R}_t(\theta^t)$, then given the previous choice of capital and human capital $\left(\tilde{k}(\hat{\theta}^{t-1}), \tilde{h}(\hat{\theta}^{t-1})\right)$, the agent of type $\theta^t \in \Theta^t$ chooses allocations $\{\tilde{c}, \tilde{z}, \tilde{y}, \tilde{h}, \tilde{k}\}$ to maximize the following life-time utility

$$\tilde{U}^t\left(\tilde{k}(\hat{\theta}^{t-1}), \tilde{h}(\hat{\theta}^{t-1}), \theta_t\right) = \max u(\tilde{c}) - \phi\left(\frac{\tilde{z}}{\tilde{h}(\hat{\theta}^{t-1})}\right) + \beta \int \tilde{U}^{t+1}(\tilde{k}, \tilde{h}, \theta_{t+1}) \pi(\theta_{t+1}) d\theta_{t+1},$$

subject to

$$\left(1 - \tilde{\tau}_z(\hat{\theta}^t)\right) w_t \tilde{z} + R_t \left(1 - \tilde{\tau}_k(\hat{\theta}^{t-1})\right) \tilde{k}\left(\hat{\theta}^{t-1}\right) - \tilde{c} - x(\hat{\theta}^t) - \tilde{y} - \tilde{k} - \Gamma(\hat{\theta}^t) \geq 0, \tag{31a}$$

$$\tilde{h} = \psi\left(x(\hat{\theta}^t), \tilde{y}\right) + \theta_t, \tag{31b}$$

$$S\left(x(\hat{\theta}^t), \tilde{z}, \tilde{k}\right) = \left(1 - \tilde{\tau}_z(\hat{\theta}^t)\right) w\left(\tilde{z} - z(\hat{\theta}^t)\right) - (\tilde{k} - k(\hat{\theta}^t)) = 0, \tag{31c}$$

where the maximization is taken over $\{\tilde{c}, \tilde{z}, \tilde{k}, \tilde{y}, \tilde{h}\}$, and the lump-sum tax $\Gamma(\hat{\theta}^t)$ is as follows.

$$\Gamma(\hat{\theta}^t) = \left(1 - \tau_z(\hat{\theta}^t)\right) w_t z(\hat{\theta}^t) + R_t \left(1 - \tau_k(\hat{\theta}^{t-1})\right) k\left(\hat{\theta}^{t-1}\right) - c(\hat{\theta}^t) - x(\hat{\theta}^t) - y(\hat{\theta}^t) - k(\hat{\theta}^t). \tag{31d}$$

Let the multipliers respect to the constraints (31a)-(31c) be $\lambda^{\hat{\sigma}}(\theta^t)$, $\mu^{\hat{\sigma}}(\theta^t)$ and $\eta^{\hat{\sigma}}(\theta^t)$. The Hamiltonian of the above problem is

$$\begin{aligned}
\mathcal{H} &= \max U^t\left(\tilde{k}\left(\hat{\theta}^{t-1}\right), \tilde{h}\left(\hat{\theta}^{t-1}\right), \theta_t\right) \\
&= \max u(\tilde{c}) - \phi\left(\frac{\tilde{z}}{\tilde{h}(\hat{\theta}^{t-1})}\right) + \beta \int U^{t+1}\left(\tilde{k}, \tilde{h}, \theta_{t+1}\right) \pi\left(\theta_{t+1}\right) d\theta_{t+1} \\
&+ \lambda^{\hat{\sigma}}\left(\theta^t\right)\left[\left(1 - \tilde{\tau}_z(\hat{\theta}^t)\right) w_t \tilde{z} + R_t \left(1 - \tilde{\tau}_k(\hat{\theta}^{t-1})\right) \tilde{k}\left(\hat{\theta}^t\right) - \tilde{c} - \tilde{x}(\hat{\theta}^t) - \tilde{y} - \tilde{k} - \Gamma_t(\hat{\theta}^t)\right] \\
&+ \mu^{\hat{\sigma}}\left(\theta^t\right)\left[\psi\left(\tilde{x}\left(\hat{\theta}^t\right), \tilde{y}\right) + \theta_t - \tilde{h}\right] \\
&+ \eta^{\hat{\sigma}}\left(\theta^t\right)\left[\left(1 - \tau_z\left(\hat{\theta}^t\right)\right) w_t \left(\tilde{z} - z\left(\hat{\theta}^t\right)\right) - \left(\tilde{k} - k\left(\hat{\theta}^t\right)\right)\right].
\end{aligned}$$

The first-order conditions with respect to $\{\tilde{c}, \tilde{z}, \tilde{k}, \tilde{y}, \tilde{h}\}$ are

$$\frac{\partial \mathcal{H}}{\partial \tilde{c}} = u'(\tilde{c}) - \lambda^{\hat{\sigma}}\left(\theta^t\right) = 0,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{z}} = -\phi'\left(\frac{\tilde{z}}{\tilde{h}(\hat{\theta}^{t-1})}\right) \frac{1}{\tilde{h}(\hat{\theta}^{t-1})} + w\left(1 - \tau_{z_1}\left(\hat{\theta}^t\right)\right)\left(\lambda^{\hat{\sigma}}\left(\theta^t\right) + \eta^{\hat{\sigma}}\left(\theta^t\right)\right) = 0,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{k}} = \beta \int U_k^{t+1}\left(\tilde{k}, \tilde{h}, \theta_{t+1}\right) \pi\left(\theta_{t+1}\right) d\theta_{t+1} - \left(\lambda^{\hat{\sigma}}\left(\theta^t\right) + \eta^{\hat{\sigma}}\left(\theta^t\right)\right) = 0,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{y}} = -\lambda^{\hat{\sigma}}\left(\theta^t\right) + \mu^{\hat{\sigma}}\left(\theta^t\right) \psi_y\left(x\left(\hat{\theta}^t\right), \tilde{y}\right) = 0,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{h}} = \beta \int U_h^{t+1}\left(\tilde{k}, \tilde{h}, \theta_{t+1}\right) \pi\left(\theta_{t+1}\right) d\theta_{t+1} - \mu^{\hat{\sigma}}\left(\theta^t\right) = 0.$$

Moreover, the envelope conditions are

$$U_k^t\left(\tilde{k}\left(\hat{\theta}^{t-1}\right),\tilde{h}\left(\hat{\theta}^{t-1}\right),\theta_t\right)=\lambda^{\hat{\sigma}}\left(\theta^t\right)R_t\left(1-\tilde{\tau}_k\left(\hat{\theta}^{t-1}\right)\right),$$

$$U_h^t\left(\tilde{k}(\hat{\theta}^{t-1}),\tilde{h}(\hat{\theta}^{t-1}),\theta_t\right)=\phi'\left(\frac{\tilde{z}}{\tilde{h}\left(\hat{\theta}^{t-1}\right)}\right)\frac{\tilde{z}}{\left[\tilde{h}\left(\hat{\theta}^{t-1}\right)\right]^2}.$$

Shifting backward by one period, these envelope conditions are

$$U_k^{t+1}\left(\tilde{k},\tilde{h},\theta_{t+1}\right)=\lambda^{\hat{\sigma}}\left(\theta^{t+1}\right)R_{t+1}\left(1-\tilde{\tau}_k(\hat{\theta}^t)\right),$$

$$U_h^{t+1}\left(\tilde{k},\tilde{h},\theta_{t+1}\right)=\phi'\left(\frac{\tilde{z}(\theta_{t+1})}{\tilde{h}}\right)\frac{\tilde{z}(\theta_{t+1})}{\left[\tilde{h}\right]^2}.$$

Besides, equation (13b) can be easily derived from the envelope condition and the first-order conditions respect to $\tilde{c}$ and $\tilde{k}$.

Also, because the preference is concave and the constraint set is a convex set, the first-order conditions are both necessary and sufficient for the maximum, and thus there is a unique solution $\{\tilde{c},\tilde{z},\tilde{y},\tilde{h},\tilde{k}\}$ to this problem. In combination with the constraints that hold with an equality, these first-order conditions and envelope conditions above give the following conditions for any $s\geq t$.

$$\frac{\phi'\left(\frac{\tilde{z}}{\tilde{h}\left(\hat{\theta}^{s-1}\right)}\right)\frac{1}{\tilde{h}\left(\hat{\theta}^{s-1}\right)}}{w_s\left(1-\tilde{\tau}_z\left(\hat{\theta}^s\right)\right)}=\beta R_{s+1}\left(1-\tilde{\tau}_k\left(\hat{\theta}^s\right)\right)\int u'\left(\tilde{c}\left(\theta_{s+1}\right)\right)\pi\left(\theta_{s+1}\right)d\theta_{s+1}, \tag{32a}$$

$$u(\tilde{c})=\beta\psi_y\left(x\left(\hat{\theta}^s\right),\tilde{y}\right)\int\phi'\left(\frac{\tilde{z}(\theta_{s+1})}{\tilde{h}}\right)\frac{\tilde{z}(\theta_{s+1})}{\left[\tilde{h}\right]^2}\pi\left(\theta_{s+1}\right)d\theta_{s+1}, \tag{32b}$$

$$\tilde{h}=\psi\left(x(\hat{\theta}^s),y\right)+\theta_s, \tag{32c}$$

$$\left(1-\tilde{\tau}_z(\hat{\theta}^s)\right)w\tilde{z}+R\left(1-\tilde{\tau}_k(\hat{\theta}^{s-1})\right)\tilde{k}(\hat{\theta}^{s-1})-\tilde{c}-x(\hat{\theta}^s)-\tilde{y}-\tilde{k}-\Gamma(\hat{\theta}^s)=0, \tag{32d}$$

$$\left(1-\tilde{\tau}_z(\hat{\theta}^s)\right)w\left(\tilde{z}-z(\hat{\theta}^s)\right)-\left(\tilde{k}-k(\hat{\theta}^s)\right)=0. \tag{32e}$$

Suppose that the linear tax rates $\{\tilde{\tau}_k(\hat{\theta}^s),\tilde{\tau}_z(\hat{\theta}^s)\}$ are set as the wedges defined in (12a)-(12b). Next, we show that, for the agent whose reporting strategy is $\hat{\sigma}$, if the resulting constrained efficient allocation $\{c^{\hat{\sigma}}(\theta^s),y^{\hat{\sigma}}(\theta^s),k(\hat{\theta}^s),z(\hat{\theta}^s),h^{\hat{\sigma}}(\theta^s)\}_{s\geq t}$ satisfies all (32a)-(32e), then the allocation must be the unique solution of the above decentralized problem.

From the definitions in (12a)-(12b) and based on Proposition 6, we can easily verify that the tuple $\{z(\hat{\theta}^s),c^{\hat{\sigma}}(\theta^{s+1})\}$ satisfies (32a). Also, based on the Proposition 5, the tuple $\{c^{\hat{\sigma}}(\theta^s),\ y^{\hat{\sigma}}(\theta^s),\ h^{\hat{\sigma}}(\theta^s),$ $z(\hat{\theta}^{s+1})\}$ satisfies (32b). Besides, based on (9), we know that the tuple $\{y^{\hat{\sigma}}(\theta^s),h^{\hat{\sigma}}(\theta^s)\}$ satisfies (32c), and based on the definition of $\Gamma(\hat{\theta}^s)$ in (31d) and the fact that $c^{\hat{\sigma}}(\theta^s)+y^{\hat{\sigma}}(\theta^s)=c(\hat{\theta}^s)+y(\hat{\theta}^s)$, the tuple $\{c^{\hat{\sigma}}(\theta^s),y^{\hat{\sigma}}(\theta^s),k(\hat{\theta}^s),z(\hat{\theta}^s)\}$ satisfies (32d). Finally it is obvious that the tuple $\{z(\theta^s),k(\theta^s)\}$ satisfies restricted condition (32e). Therefore, for the agent whose reporting strategy is $\hat{\sigma}$, the constrained efficient allocation indeed satisfies all (32a)-(32e), which implies that conditions (32a)-(32e) yield the allocations that are the same as the constrained efficient allocation $\{c^{\hat{\sigma}}(\theta^s),\ y^{\hat{\sigma}}(\theta^s),\ k(\hat{\theta}^s),\ z(\hat{\theta}^s),\ h^{\hat{\sigma}}(\theta^s)\}$.

In addition, the incentive compatibility constraint (11b) implies that when type θ_t agents choose a truth-telling strategy, i.e., $\hat{\theta}_t = \theta_t$, the constrained efficient allocation $\{c^{\hat{\sigma}}(\theta^s),\ y^{\hat{\sigma}}(\theta^s),\ k(\hat{\theta}^s),\ z(\hat{\theta}^s),\ h^{\hat{\sigma}}(\theta^s)\}$ gives the highest lifetime utility. Thus, the best strategy is to report the type truthfully, and the constrained efficient allocation $\{c^{\hat{\sigma}}(\theta^s), y^{\hat{\sigma}}(\theta^s), k(\hat{\theta}^s), z(\hat{\theta}^s), h^{\hat{\sigma}}(\theta^s)\}$ indeed solves the utility maximization problem for type θ_t agents under this tax system. Therefore, our tax system indeed implements the constrained efficient allocation in an equilibrium. □

A.10 Tax Implementation in the Two-period Model in Subsection 5.3

In Subsection 5.4, the problem of an agent of type θ is to maximize

$$\max u\left(\tilde{c}_1(\theta)\right) - \phi\left(\tfrac{\tilde{z}_1(\theta)}{h_1}\right) + \beta\left[u\left(\tilde{c}_2(\theta)\right) - \phi\left(\tfrac{\tilde{z}_2(\theta)}{\tilde{h}_2(\theta)}\right)\right],$$

subject to budget constraints and human capital accumulation as follows.

$$\tilde{c}_1(\theta) + \tilde{x}_1(\theta) + \tilde{y}_1(\theta) + \tilde{k}_2(\theta) \le w_1\tilde{z}_1(\theta) + (1+r_1)k_1 - T_1,$$

$$\tilde{c}_2(\theta) \le w_2\tilde{z}_2(\theta) + (1+r_2)\tilde{k}_2(\theta) - T_2,$$

$$\tilde{h}_2(\theta) = (1-\delta_h)h_1 + \psi\left(\tilde{x}_1(\theta), \tilde{y}_1(\theta)\right) + \theta,$$

where the maximization is made over $\{\tilde{c}_1(\theta), \tilde{c}_2(\theta), \tilde{z}_1(\theta), \tilde{z}_2(\theta), \tilde{x}_1(\theta), \tilde{y}_1(\theta), \tilde{h}_2(\theta), \tilde{k}_2(\theta)\}$.

By only restricting $x_1 = x_1(\sigma)$ for some $\sigma \in \Theta$, the agent with true type $\theta \in \Theta$ chooses allocations $\{c_1, c_2, y_1, h_2, z_1.z_2, k_2\}$ to maximize $u(c_1) - \phi\left(\frac{z_1}{h_1}\right) + \beta\left[u(c_2) - \phi\left(\frac{z_2}{h_2}\right)\right]$, subject to

$$\Gamma_1(\sigma) + \left(1-\tau_{z_1}(\sigma)\right)w_1z_1 + \left(1-\tau_{k_1}(\sigma)\right)\left(1+r_1\right)k_1 - c_1 - x_1(\sigma) - y_1 - k_2 \ge 0,$$

$$\Gamma_2(\sigma) + \left(1-\tau_{z_2}(\sigma)\right)w_2z_2 + \left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)k_2 - c_2 \ge 0,$$

$$h_2 = (1-\delta_h)h_1 + \psi\left(x_1(\sigma), y_1\right) + \theta.$$

From the resulting first-order conditions, we get

$$u'(c_1) = \beta(1+r_2)\left(1-\tau_{k_2}(\sigma)\right)u'(c_2)$$

However, given the reporting type $\sigma \in \Theta$, the efficient constrained allocation of consumption in the first period $c_1^{\sigma}(\theta)$ may vary with true type $\theta \in \Theta$, therefore, it is impossible to find a capital tax rate $\tau_{k_2}(\sigma)$ such that the efficient constrained allocation $\{c_1^{\sigma}(\theta),\ c_2(\sigma)\}$ satisfy the above equation, especially when $\theta \ne \sigma$, which means that the allocation in the competitive equilibrium are different from the efficient constrained allocation. If we add in the following two constraints,

$$S_1^{\sigma}(z_1, k_2) = \left(1-\tau_{z_1}(\sigma)\right)w_1(z_1 - z_1(\sigma)) - (k_2 - k_2(\sigma)) = 0,$$

$$S_2^{\sigma}(c_2, k_2) = \left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)(k_2 - k_2(\sigma)) - \left(c_2 - c_2(\sigma)\right) = 0,$$

then, the resulting allocations in the competitive equilibrium are the same as the efficient constrained allocation with following lump-sum taxes:

$$\Gamma_1(\sigma)=c_1(\sigma)+x_1(\sigma)+y_1(\sigma)+k_2(\sigma)-\left(1-\tau_{z_1}(\sigma)\right)w_1z_1(\sigma)+\left(1+r_1\right)k_1, \tag{33a}$$

$$\Gamma_2(\sigma)=c_2(\sigma)-\left(1-\tau_{z_2}(\sigma)\right)w_2z_2(\sigma)+\left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)k_2(\sigma). \tag{33b}$$

Specifically, the Lagrangian is

$$\begin{aligned}\mathcal{L}=&\max_{c_1,c_2,y_1,h_2,z_1,z_2,k_2} u(c_1)-\phi\left(\tfrac{z_1}{h_1}\right)+\beta\left[u(c_2)-\phi\left(\tfrac{z_2}{h_2}\right)\right]\\&+\lambda_1'\left[\Gamma_1(\sigma)+\left(1-\tau_{z_1}(\sigma)\right)w_1z_1+\left(1-\tau_{k_1}(\sigma)\right)\left(1+r_1\right)k_1-c_1-x_1(\sigma)-y_1-k_2\right]\\&+\lambda_2'\left[\Gamma_2(\sigma)+\left(1-\tau_{z_2}(\sigma)\right)w_2z_2+\left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)k_2-c_2\right]\\&+\mu'\left[\left(1-\delta_h\right)h_1+\psi\left(x_1(\sigma),y_1\right)+\theta-h_2\right]\\&+\eta_1'\left[w_1\left(z_1-z_1(\sigma)\right)\left(1-\tau_{z,1}(\sigma)\right)-\left(k_2-k_2(\sigma)\right)\right]\\&+\eta_2'\left[\left(c_2-c_2(\sigma)\right)-\left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)\left(k_2-k_2(\sigma)\right)\right].\end{aligned}$$

The first-order conditions are

$$\frac{\partial\mathcal{L}}{\partial c_1}=u'(c_1)-\lambda_1'=0,$$

$$\frac{\partial\mathcal{L}}{\partial c_2}=\beta u'(c_2)-\lambda_2'+\eta_2'=0,$$

$$\frac{\partial\mathcal{L}}{\partial z_1}=-\phi'\left(\tfrac{z_1}{h_1}\right)\tfrac{1}{h_1}+w_1\left(1-\tau_{z_1}(\sigma)\right)\left(\lambda_1'-\eta_1'\right)=0,$$

$$\frac{\partial\mathcal{L}}{\partial z_2}=\beta\phi'\left(\tfrac{z_2}{h_2}\right)\tfrac{1}{h_2}-\lambda_2'w_2\left(1-\tau_{z_2}(\sigma)\right)=0,$$

$$\frac{\partial\mathcal{L}}{\partial k_2}=-\lambda_1'+\eta_1'+\left(\lambda_2'-\eta_2\right)\left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)=0,$$

$$\frac{\partial\mathcal{L}}{\partial y_1}=-\lambda_1+\mu'\psi_y\left(x_1(\sigma),y_1\right)=0,$$

$$\frac{\partial\mathcal{L}}{\partial h_2}=\beta\phi'\left(\tfrac{z_2}{h_2}\right)\tfrac{z_2}{(h_2)^2}-\mu'=0.$$

From the first-order conditions we obtain the following (34a)-(34b). Combined with the constraints written as equalities, we list all the conditions of this problems as follows:

$$\phi'\left(\tfrac{z_1}{h_1}\right)\tfrac{1}{h_1}=\beta u'(c_2)\left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)w_1\left(1-\tau_{z_1}(\sigma)\right), \tag{34a}$$

$$u'(c_1)=\beta\phi'\left(\tfrac{z_2}{h_2}\right)\tfrac{z_2}{(h_2)^2}\psi_y\left(x_1(\sigma),y_1\right), \tag{34b}$$

$$\Gamma_1(\sigma)+\left(1-\tau_{z_1}(\sigma)\right)w_1z_1+\left(1-\tau_{k_1}(\sigma)\right)\left(1+r_1\right)k_1-c_1-x_1(\sigma)-y_1-k_2=0, \tag{34c}$$

$$\Gamma_2(\sigma)+\left(1-\tau_{z_2}(\sigma)\right)w_2z_2+\left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)k_2-c_2=0, \tag{34d}$$

$$h_2=\left(1-\delta_h\right)h_1+\psi\left(x_1(\sigma),y_1\right)+\theta, \tag{34e}$$

$$S_1^{\sigma} = w_1\left(z_1 - z_1(\sigma)\right)\left(1-\tau_{z,1}(\sigma)\right) + \left(k_2 - k_2(\sigma)\right) = 0, \tag{34f}$$

$$S_2^{\sigma} = \left(c_2 - c_2(\sigma)\right) - \left(1-\tau_{k_2}(\sigma)\right)\left(1+r_2\right)\left(k_2 - k_2(\sigma)\right) = 0. \tag{34g}$$

Although these conditions (34a)-(34g) are necessary and not sufficient, it is noted that our problem uses a quadratic objective function with linear constraints. Then, these conditions (34a)-(34g) are also sufficient, if the Jacobean matrix of the first-order conditions is negative semi-definite. The Jacobean matrix, $\nabla^2\mathcal{L}$, is as follows.

$$\nabla^2\mathcal{L}. = \begin{bmatrix} u''(c_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta u''(c_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\phi''\left(\frac{z_1}{h_1}\right)\frac{1}{(h_1)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta\phi''\left(\frac{z_2}{h_2}\right)\frac{1}{(h_2)^2} & 0 & 0 & \Psi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu'\psi_{yy}\left(x_1(\sigma), y_1\right) & 0 \\ 0 & 0 & 0 & \Psi & 0 & 0 & \Sigma \end{bmatrix}$$

where $\Psi = \beta[\phi''\left(\frac{z_2}{h_2}\right)\frac{z_2}{(h_2)^3} + \phi'\left(\frac{z_2}{h_2}\right)\frac{1}{(h_2)^2}]$ and $\Sigma = -\beta[\phi''\left(\frac{z_2}{h_2}\right)\frac{(z_2)^2}{(h_2)^4} + 2\phi'\left(\frac{z_2}{h_2}\right)\frac{z_2}{(h_2)^3}]$.

Obviously, the matrix is negative semi-definite, since one of the columns has all elements equal to zero. Hence, there is a unique solution to the above problem. With the Inada condition, the solution is interior. Therefore, the first-order conditions are both necessary and sufficient for the maximum.

Now, we show that, for agents with type θ who choose strategy σ if their constrained efficient allocation $\{c_1^{\sigma}(\theta),\ c_2(\sigma),\ y_1^{\sigma}(\theta),\ k_2(\sigma),\ z_1(\sigma),\ z_2(\sigma),\ h_2^{\sigma}(\theta)\}$ satisfies all (34a)-(34g), then the allocation must be the unique solution of the above decentralized problem.

For any $\sigma \in \Theta$, from the definitions in (7a)-(7b) and the condition in (33a), one can easily verify that tuple $\{z_1(\sigma), c_2(\sigma)\}$ satisfies (34a),. Also, based on the Proposition 1, the tuple $\{c_1^{\sigma}(\theta),\ y_1^{\sigma}(\theta),\ z_2(\sigma),\ h_2^{\sigma}(\theta)\}$ satisfies (34b). Besides, based on (33a) and (1c), we know that in the constrained efficient allocation *A*, the tuple $\{c_1^{\sigma}(\theta), y_1^{\sigma}(\theta), k_2(\sigma), z_1(\sigma),\}$ satisfies (34c), and based on (33b), the tuple $\{\ c_1^{\sigma}(\theta), c_2(\sigma), k_2(\sigma), z_2(\sigma),\ \}$ satisfies (34d) as well. Moreover, according to (1d), the tuple $\{y_1^{\sigma}(\theta), h_2^{\sigma}(\theta)\}$ satisfies (34e). Finally it is obviously that the tuple $\{z_1(\sigma), k_2(\sigma), c_2(\sigma)\}$ satisfy strategy restrictions, (34f) and (34g). Therefore, the constrained efficient allocation for the agent whose reporting strategy is $\sigma(\theta)$, indeed satisfies all (34a)-(34g), which implies that these conditions (34a)-(34g) yield allocations that are the same as the efficient constrained allocations $\{c_1^{\sigma}(\theta),\ c_2(\sigma),\ y_1^{\sigma}(\theta),\ k_2(\sigma),\ z_1(\sigma),\ z_2(\sigma),\ h_2^{\sigma}(\theta)\}$.

Thus, when these two constraints are met, the following tax system implements constrained efficient allocations; namely, in the first period, the taxes are

$$T_1 = \tilde{T}_1(\theta) \equiv \Gamma_1(\theta) + \tilde{\tau}_{z_1}(\theta) w_1 z_1,$$

if there is some $\theta \in \Theta$ such that the condition $S_1^{\theta}(z_1, k_2) = 0$ holds and $x_1 = x_1(\theta)$; otherwise, $T_1 = \infty$; and in the second period, the taxes are

$$T_2 = \tilde{T}_2(\theta) \equiv \Gamma_2(\theta) + \tilde{\tau}_{z_2}(\theta) w_2 z_2 + \tilde{\tau}_{k_2}(\theta) r_2 k_2,$$

if there is some $\theta \in \Theta$ such that the condition $S_1^\theta(z_1, k_2) = S_2^\theta(k_2, c_2) = 0$ holds; otherwise, $T_2 = \infty$.

Note that, when using a tax system to implement constrained efficient allocations, the set of conditions are not unique. There are other ways. For example, if we set $S_2^\sigma = z_2 - z_2(\sigma)$, we believe that this set can also implement efficient constrained allocations, but it is too restrictive.

A.11 The Conditions in the Decentralized Problem for Calibration in Section 6

In the decentralized economy, the problem for an agent with skill type θ^t is to

$$\max u(c_t) - \phi\left(\tfrac{z_t}{h_t}\right) + \sum_{s=t+1}^{T} \beta^{s-t} E_t\left[u(c_s) - \phi\left(\tfrac{z_s}{h_s}\right)\right],$$

subject to (17a) and (17b), rewritten as follow, respectively.

$$c_s + x_s + y_s + k_{s+1} \le (1 - \tau_z^b) w_s z_s + R_s k_s + LS_s, \quad \text{for } s = t, t+1, \ldots, T,$$

$$h_{t+1} = \psi(x_t, y_t) + \theta_t.$$

The Lagragian is:

$$\mathcal{L} = \max \sum_{s=t}^{T} \beta^{s-t} E_t\left[u(c_s) - \phi\left(\tfrac{z_s}{h_s}\right)\right] + \lambda_s\left[(1 - \tau_z^b) w_s z_s + R_s k_s + LS_s - c_s - x_s - y_s - k_s\right]$$

and the first-order conditions are

$$[c_s]:\ \beta^{s-t} E_t\left[u'(c_s)\right] = \lambda_s,$$

$$[z_s]:\ \beta^{t-s} \phi'\left(\tfrac{z_s}{h_s}\right)\tfrac{1}{h_s} = \lambda_s (1 - \tau_z^b) w_s,$$

$$[x_s]:\ \beta^{s-t+1} E_t\left[\phi'\left(\tfrac{z_{s+1}}{h_{s+1}}\right)\tfrac{z_{s+1}}{[h_{s+1}]^2}\right]\psi_x(x_s, y_s) = \lambda_s,$$

$$[y_s]:\ \beta^{s-t+1} E_t\left[\phi'\left(\frac{z_{s+1}}{h_{s+1}}\right)\frac{z_{s+1}}{[h_{s+1}]^2}\right]\psi_y(x_s, y_s) = \lambda_s,$$

$$[k_{s+1}]:\ \lambda_s = E_s\left[\lambda_{s+1}(1 - \tau_k^b) R_{s+1}\right].$$

From the above conditions, we can derive the following four equations for calibration.

First, we get $u'(c_t) = \beta(1 - \tau_k^b) R_{t+1} E_t[u'(c_{t+1})]$, which implies

$$(c_t(\theta))^{-\chi} = \beta R_{t+1}(1 - \tau_k) E_t\left[(c_{t+1}(\theta))^{-\chi}\right]. \tag{35a}$$

Next, we obtain $\phi'\left(\frac{z_t}{h_t}\right)\frac{1}{h_t} = u'(c_t)(1 - \tau_z^b) w_t$, which implies

$$(1 - \tau_z) w_t (c_t(\theta))^{-\chi} = \left(\frac{z_t(\theta)}{h_t}\right)^{\gamma-1} \frac{1}{h_t}. \tag{35b}$$

Further, we find $\beta E_t\left[\phi'\left(\frac{z_{t+1}}{h_{t+1}}\right)\frac{z_{t+1}}{[h_{t+1}]^2}\right]\psi_x(x_t, y_t) = u'(c_t)$, which implies

$$\left(c_t\left(\theta^t\right)\right)^{-\chi} = \beta E_t\left[\left(\frac{z_{t+1}\left(\theta^{t+1}\right)}{h_{t+1}\left(\theta^t\right)}\right)^{\gamma}\right]\frac{B\eta\left(1-\rho\right)\left(x_t\left(\theta^t\right)\right)^{\eta(1-\rho)-1}\left(y_t\left(\theta^t\right)\right)^{\eta\rho}}{h_{t+1}\left(\theta^t\right)}. \tag{35c}$$

Finally, we have $\beta E_t\left[\phi'\left(\frac{z_{t+1}}{h_{t+1}}\right)\frac{z_{t+1}}{[h_{t+1}]^2}\right]\psi_y(x_t, y_t) = u'(c_t)$, which implies

$$\left(c_t\left(\theta^t\right)\right)^{-\chi} = \beta E_t\left[\left(\frac{z_{t+1}\left(\theta^{t+1}\right)}{h_{t+1}\left(\theta^t\right)}\right)^{\gamma}\right]\frac{B\eta\rho\left(x_t\left(\theta^t\right)\right)^{\eta(1-\rho)}\left(y_t\left(\theta^t\right)\right)^{\eta\rho-1}}{h_{t+1}\left(\theta^t\right)}. \tag{35d}$$

Along with the use of (17a)-(17b), these four equations above (35a)-(35d) are employed to solve for $\{c_t(\theta^t), z_t(\theta^t), x_t(\theta^t), y_t(\theta^t), k_{t+1}(\theta^t), h_{t+1}(\theta^t)\}_{t\leq T}$.

Table 1 Exogenously calibrated parameters

Definition	Symbol	Value	Source/Note
Population			
The lower bound of type distribution	$\underline{\theta}$	0.5	Normalization
Degree of uncertainty	$\hat{\sigma}^2$	0.0095	Farhi and Werning (2013)
Preference			
Disutility elasticity	κ	3	Farhi and Werning (2013)
Discount factor	β	0.95	Farhi and Werning (2013)
Gross interest rate	R_t	1.053	
Human capital Technology			
Human capital tech level	B	1	Normalization
Share of non-verifiable education	ρ	0.667	Ewijk and Tang (2000)
Initial human capital	h_1	1	Normalization
Wage rate	w_t	1	Normalization
Tax system			
Capital income tax rate	τ_k^b	0.3	McDaniel (2007)
Labor income tax rate	τ_z^b	0.2	McDaniel (2007)
Government expenditure	G_t	0	By Assumption

Table 2. Endogenously matched parameters

Calibrated parameter		Value	Target	Value	Source
Education degree	η	0.4	Education expense ratio	0.19	Stantcheva (2017)
Upper bound of type	$\overline{\theta}$	1.5	Wage premium	1.8	Various sources*

* Murphy and Welch (1992), Autor et al. (1998), Heathcote et al. (2005), and James (2012).

Table 3. Welfare gains over laissez-faire no-tax economy

Economies	$\hat{\sigma}^2 = 0.00625$	$\hat{\sigma}^2 = 0.0095$	$\hat{\sigma}^2 = 0.0161$
Our second-best model	1.47%	1.74%	1.96%
Simple non-linear tax	1.29%	1.55%	1.89%
As % of second-best	87.8%	89.1%	96.4%

Note: Welfare gains are in terms of consumption equivalence. Simple non-linear tax is the welfare gain of an otherwise our model except with the simple history-independent, non-linear tax policy.

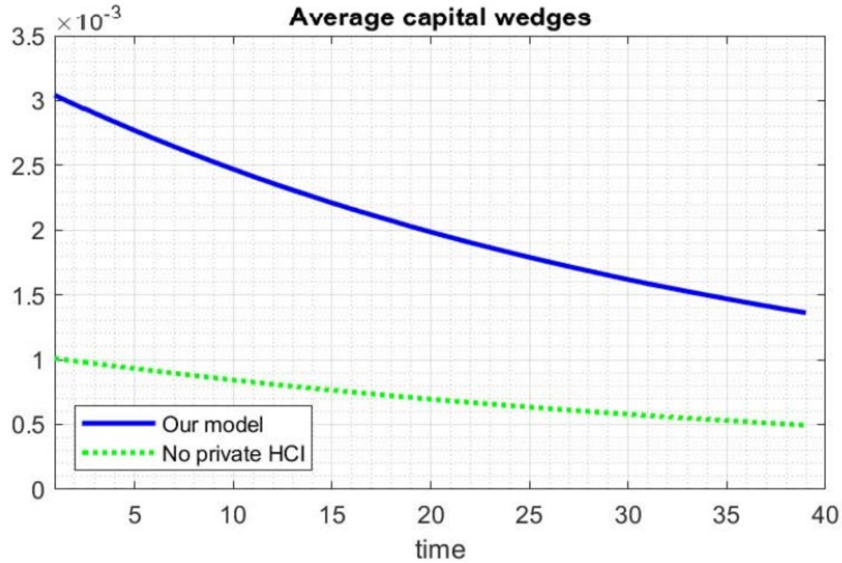


Figure 1. Average capital wedges over time

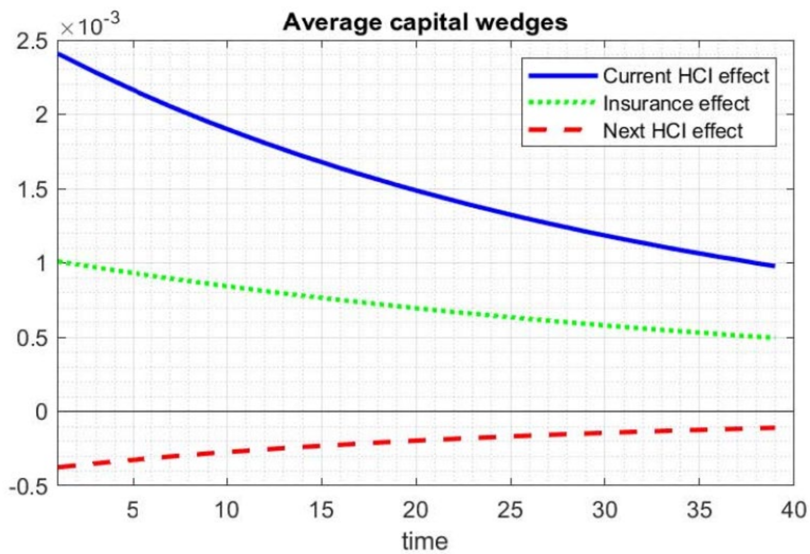


Figure 2. Decomposition of the average capital wedge into different sources of effects

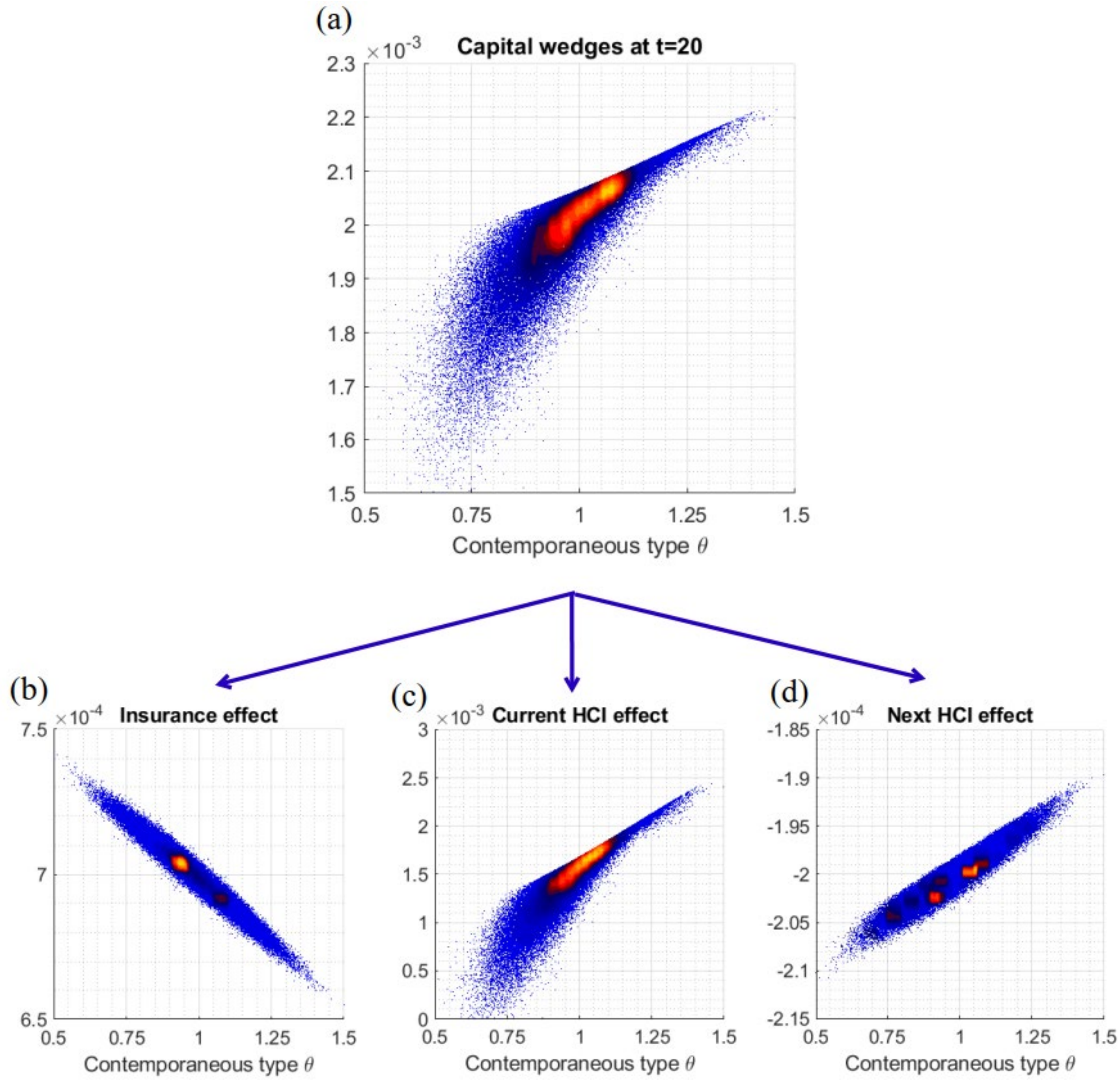


Figure 3. Scatter plot of the capital wedge against skill types at $t = 20$ and the decomposition into three sources of effects.

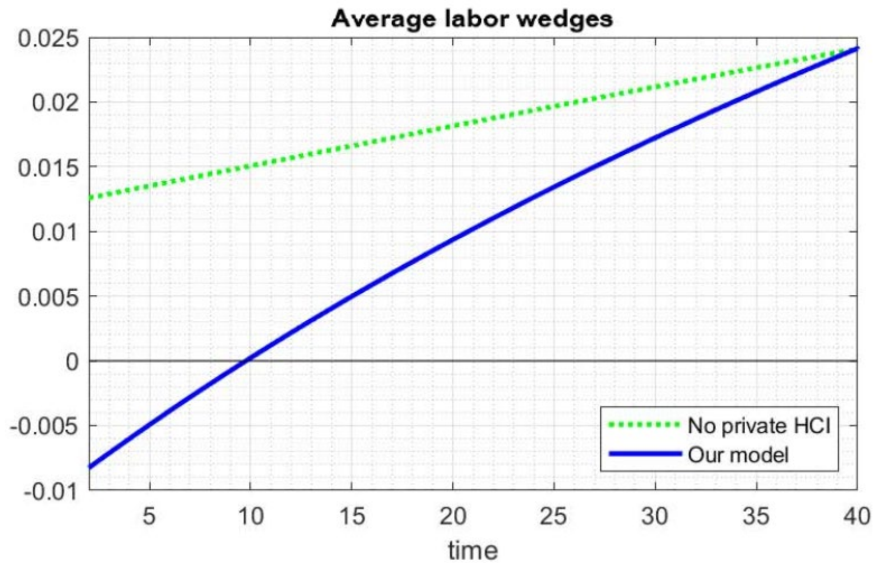


Figure 4. Average labor wedges over time.

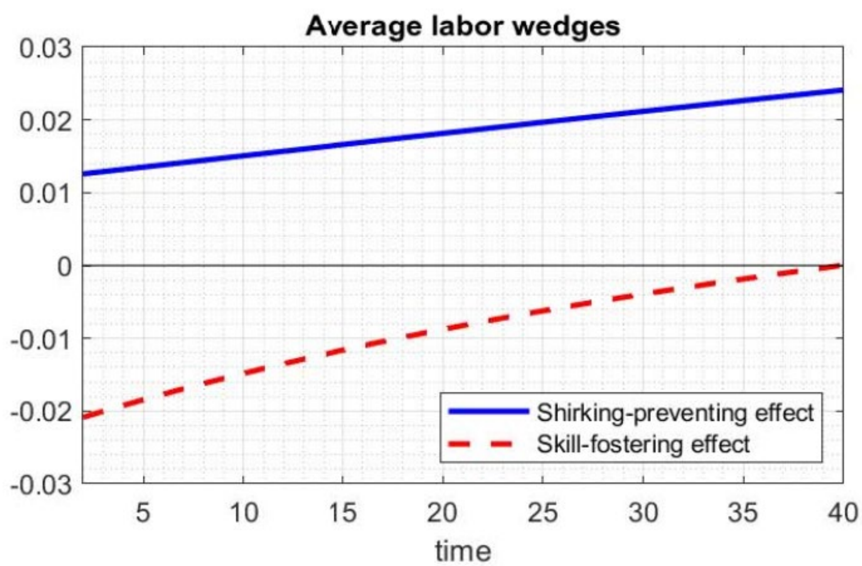


Figure 5. Decomposition of the average labor wedge into different sources of effects.

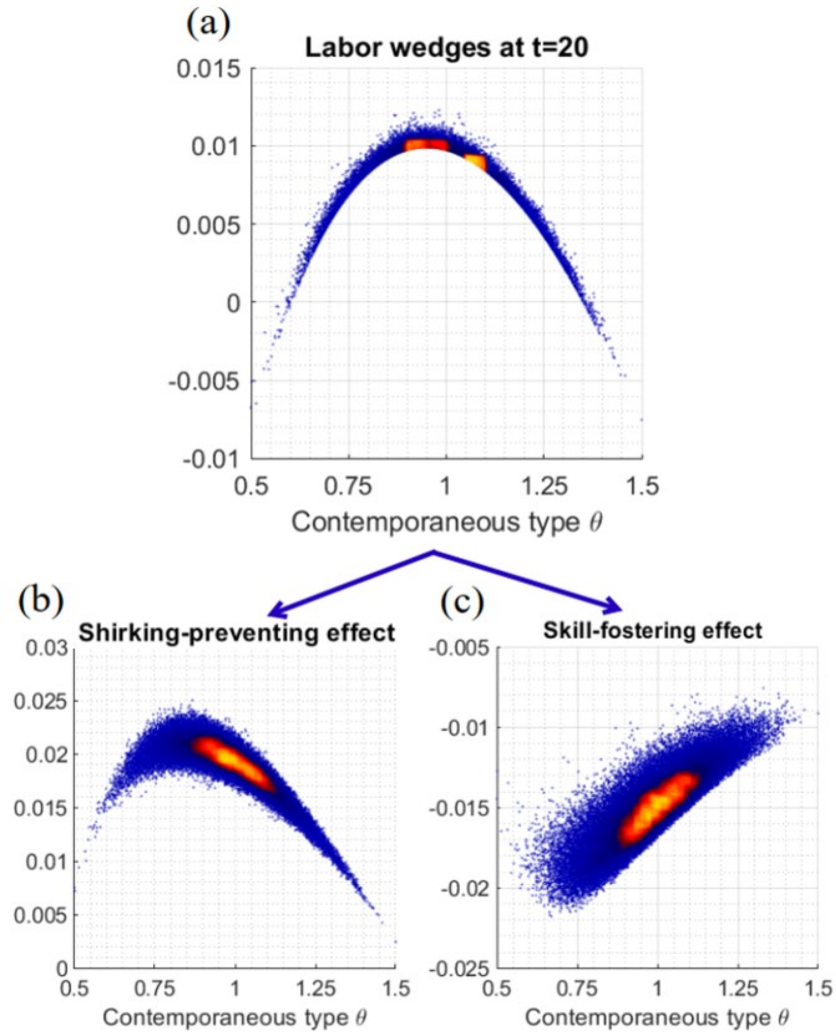
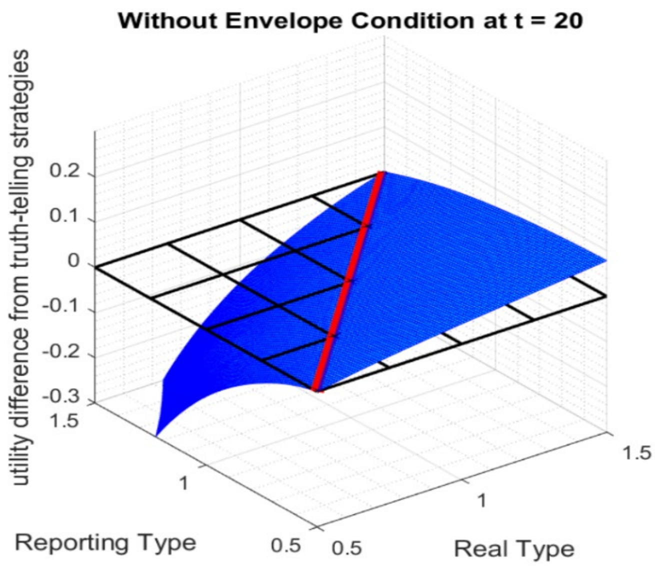
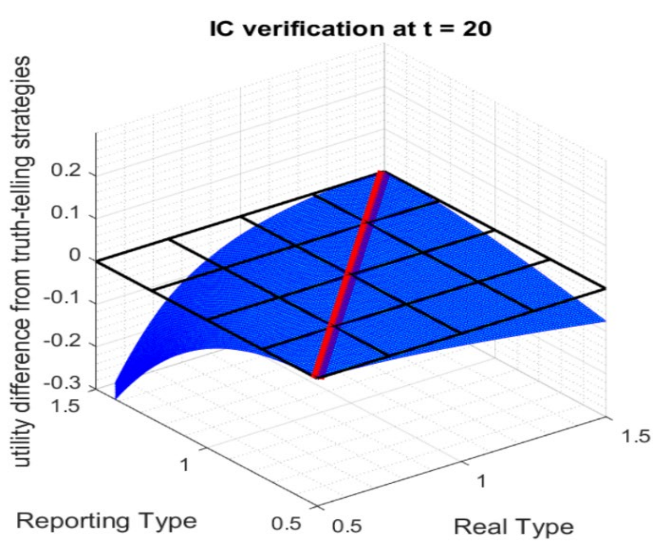


Figure 6. Scatter plot of the labor wedge against skill types at $t = 20$ and the decomposition into two sources of effects.



Note. The x-axis is the true type θ, the y-axis is the reporting type σ, and the z-axis is $W^{\sigma}(\theta) - W(\theta)$ in (10a)-(10b). In our model except for the envelope condition, we calculate the utility difference of the reporting strategies σ from the truth-telling strategy in periods 1, 2, ..., 40 over the life cycle, and this figure reports period $t = 20$ as an example.

Figure 7. Utility gains from different reporting strategies in otherwise our model except the envelope condition ($t = 20$).



Note. The x-axis, the y-axis, and the z-axis are the same as Figure 7. We calculate the utility difference of the reporting strategies σ from the truth-telling strategy in periods 1,2,..., 40 over the life cycle in our model (with the envelope condition). This figure reports $t = 20$ as an example for ex post verification.

Figure 8. Utility gains from different reporting strategies in our model ($t = 20$).