Southern Economic Journal 2009, 75(3), 639–662

Two-Sector Growth Models with Productive Public Goods: Equilibrium (In)determinacy

Been-Lon Chen* and Shun-Fa Lee†

In existing two-sector, human capital–based endogenous growth models with social constant returns, local equilibrium indeterminacy emerges based upon either differential factor tax rates or sector-specific externalities. Two primary results are established in this paper. First, once there are productive public goods, the two existing mechanisms are not robust in establishing local indeterminacy. Second, with the congestion effect in the use of public services, local indeterminacy is regained.

JEL Classification: D90, O40, O41

1. Introduction

Models of economic growth with multiple equilibrium paths offer an explanation as to why we often observe diverse growth performances among countries with similar economic environments. A central question is "Under what conditions is it possible to establish local indeterminacy?" In this paper, we study a Lucas (1988) human capital–based growth model and investigate whether there are multiple equilibrium paths and, thus, local equilibrium indeterminacy in the neighborhood of a balanced growth path.

Equilibrium indeterminacy in existing two-sector, human capital–based growth models emerges based upon two mechanisms. One of the mechanisms is differential factor tax rates, initiated in the work of Bond, Wang, and Yip (1996) and followed by that of Raurich (2001). The other mechanism is sector-specific externalities, pioneered in the work of Benhabib and Farmer (1996) in an exogenous growth model and followed by the work of Benhabib, Meng, and Nishimura (2000) and Mino (2001) in endogenous growth models. Ben-Gad (2003) combined the two mechanisms. These two mechanisms establish local indeterminacy when the mechanisms cause the factor intensities at the social perspective and those at the private perspective to reverse.

The purpose of this paper is twofold. First, we argue that once productive public goods are considered, the two existing mechanisms are not robust in creating local indeterminacy. Then we take into account congestion in public services, so a two-sector model with the two existing mechanisms can generate local indeterminacy. Our purpose is motivated by the following.

---

* Institute of Economics, Academia Sinica, 128 Academia Road, Section 2, Taipei 11529, Taiwan; E-mail bchen@econ.sinica.edu.tw; corresponding author.

† Department of Industrial Economics, Tamkang University, 151 Ying-chuan Road, Tamsui, Taipei County 25137, Taiwan; E-mail alphalee@mail.tku.edu.tw.

We are grateful to two anonymous referees and an editor for constructive comments and to participants in several workshops for valuable comments and suggestions in earlier versions of the paper.

Received May 2007; accepted December 2007.

In the existing models with sector-specific externalities or distorted factor taxes, there is either no government spending or, if the government spending is taken into account, only consumptive public spending and government transfers are considered. Recently, a number of authors have documented the positive impact of public expenditure upon the productivity of private capital.[1] When there is productive public spending, the production technology is externally enhanced by the public goods. Productive public goods as an engine of economic growth have received much attention in endogenous growth models. In particular, a number of studies in this line of modeling strategy have obtained local indeterminacy.[2] Thus, it is important to determine whether or not differential factor tax rates and sector-specific externalities create local indeterminacy in models with productive public goods.

However, as we will find, the two existing mechanisms no longer create local indeterminacy in a two-sector growth model with productive public goods unless the difference in the factor tax rates and the gap in the sector-specific externalities are implausibly large. Intuitively, productive public goods introduce the effects of intersectoral externalities. They offset the effects of the two existing mechanisms that would have otherwise reversed the factor intensity at the social viewpoint from the factor intensity at the private viewpoint. As a consequence, the two existing mechanisms are not robust in the establishment of local indeterminacy in models with productive public goods.

This paper, then, introduces the congestion effect in the use of public goods, an effect offsetting the effect from public spending, to salvage the two existing mechanisms.[3] Congestion is a prevalent experience in the use of public services, such as irrigation systems, highways, harbors, airports, and other facilities. Such an effect has received much attention in public and spatial economics (e.g., Starrett 1988; Edwards 1990). The congestion effect has recently been taken into account extensively in endogenous growth models (e.g., Barro and Sala-i-Martin 1992; Glomm and Ravikumar 1997; Fisher and Turnovsky 1998).

With a congestion effect in the use of public goods, we find that local indeterminacy arises. Intuition suggests that under constant returns, the congestion effect reduces an otherwise strong intersectoral external effect as a result of the marginal contribution of public goods services and increases the marginal contribution of aggregate physical, as well as human, capital. This makes marginal social products deviate from the marginal private products. The otherwise negative sloping relative marginal products of labor between the good and the human capital sectors are now positive sloping. As a result, a higher expected price in the sector that is

---

[1] In a study using data from the United States, Aschauer (1989) has documented positive impacts of public expenditure on the productivity of private capital and has generated extensive research efforts aimed at determining the robustness of a positive effect. Lynde and Richmond (1993) found that public capital has played an essential role in enhancing the productivity growth of United Kingdom manufacturing. See Gramlich (1994) for a survey of this empirical literature, most of which is for the United States.

[2] Endogenous growth models with productive public goods were initiated in Barro (1990). Futagami, Morita, and Shibata (1993) modified the original Barro model by assuming accumulated government services. Several studies found local indeterminacy by extending the original Barro model. For example, Cazzavillan (1996) and Chen (2006) assumed a felicity that exhibits increasing returns in consumption and public consumption; Palivos, Yip, and Zhang (2003), Raurich (2003), and Chen (2007) added utility of leisure; Park and Philippopoulos (2004) assumed monopolistic competition in the goods market; and Hu, Ohdori, and Shimomura (2005) included both pure consumption goods and pure investment goods.

[3] In a one-sector AK model, Kehoe (1991, pp. 2133–4) presents an example with local indeterminacy when physical capital exhibits negative externalities. His argument relies upon a quadratic production (in k and K) and fails to survive if a Cobb-Douglas technology is used.

more capital intensive *reduces*, *rather than increases*, the fractions of capital and labor allocated to that sector, thereby making the expectations about the higher price self-fulfilling.

Elsewhere, it was shown that congestible public goods can generate indeterminacy in a two-sector, constant-return human capital–enhanced growth model (Chen and Lee 2007). Unlike the work of Chen and Lee (2007), the current paper is an attempt to show that the arguments based on the mechanisms of sector-specific externalities and factor taxes that would have otherwise established indeterminacy are not robust and become fragile once there are productive public goods. In an effort to salvage these two popular, existing mechanisms, in the end of the paper we will follow Chen and Lee (2007) and introduce the congestion mechanism. As expected, indeterminacy is regained in the model with sector-specific externalities and factor taxes.

As developed below, section 2 sets up a model that features factor taxation, sector-specific externalities, and productive public goods, while section 3 studies the optimization and the *Balanced Growth Path* (BGP) equilibrium. Section 4 intuitively explains the reasons underlying determinacy and indeterminacy of a BGP in different set-ups, and the transitional dynamics are analyzed in section 5, in which the first subsection notes economies without congestion effects and the subsequent subsection discusses economies with congestion effects. Finally, some concluding remarks are made in section 6.

2. The Model

The model is based upon the work of Lucas (1988), Rebelo (1991), Bond, Wang, and Yip (1996), and Ben-Gad (2003) and is general enough to include both the features of the sector-specific externalities and the distortionary factor taxation in existing literature. The economy consists of a continuum of representative consumers/producers with two sectors, referred to as the goods sector, Y, and the education sector, X. Human and physical capital are both used in the two sectors, with their productivity enhanced by public good services à la Barro (1990).

More specifically, we assume both sectors are of constant returns to scale and, for simplicity, of the following parametric forms:

$$Y = A(vk)^{\alpha}(uh)^{\beta}\left(\overline{uH}\right)^{\varepsilon}\tilde{G}^{\gamma}, \quad K(0) > 0, H(0) > 0, \tag{1}$$

given

$$X = B[(1-v)k]^{\eta}[(1-u)h]^{\theta}\left[\overline{(1-u)H}\right]^{\xi}\tilde{G}^{\delta}, \tag{2}$$

in which variables with an upper bar indicate externalities and where $0 < v < 1$ (resp. $0 < u < 1$) is the fraction of physical capital, k (resp. human capital, h), allocated to the goods sector. Parameters $A > 0$ and $B > 0$ are productivity coefficients; $0 < \alpha < 1$ ($0 < \eta < 1$) and $0 < \beta < 1$ ($0 < \theta < 1$) are the shares of physical and human capital in the goods (education) sector. Moreover, both sectors are subject to sector-specific externalities, with the productivity in each sector enhanced by the average amount of aggregate human capital (denoted as H) employed in each sector, and the contribution of externality in the goods (education) sector being $\varepsilon > 0$ ($\xi > 0$). Finally, the public services, $\tilde{G}$, affect both sectors, with parameter γ (δ) representing the degree to which the public services enhance the productivity in the goods (education) sector.

Let us comment on the sector-specific externalities generated only by human capital. In general, physical and/or human capital may create sector-specific externalities (e.g., Benhabib, Meng, and Nishimura 2000; Mino 2001). The appearance of only one source of sector-specific externalities in the basic model is to maintain the insights into the existing models with sector-specific externalities while making the model as simple as possible in order to be easily tractable. In Appendix D, we have considered sector-specific externalities of both physical capital and human capital in the technology. As the Appendix shows, the dynamic equilibrium system is exactly the same as the model using the technology in Equations 1 and 2. As a result, the model with two sources of sector-specific externalities is more complicated, but the main results are expected to be the same as those in the model with one source of sector-specific externalities. We thus present in the text the model with one source of sector-specific externalities in order to keep the model simple, while leaving the model with two sources of sector-specific externalities in the Appendix for references.

There is a congestion effect in that the amount of public goods services received by the representative agent in Equations 1 and 2 may be less than the actual amount of public spending, G, with the perceived amount decreasing in the aggregate activity. Specifically, we assume that public services received by the representative agent are as follows:

$$\tilde{G} = G \cdot \left(\frac{1}{K}\right)^{\psi} \left(\frac{1}{H}\right)^{\zeta}, \tag{3}$$

in which K is aggregate physical capital, ψ indicates the degree of congestion due to the aggregate use of physical capital, and ζ is the degree of congestion due to the aggregate use of human capital. If $\psi = \zeta = 0$, the public service is non-rival and non-excludable and is therefore a pure public good.

The following parameter restrictions are imposed for the technologies:

ASSUMPTION 1

(i) $\alpha - \gamma\psi > 0$, $\beta + \varepsilon - \gamma\zeta > 0$, $\eta - \delta\psi > 0$, and $\theta + \delta\xi > 0$;
(ii) $\alpha + \beta + \varepsilon - \gamma(\psi + \zeta) = 1 - \gamma$ and $\eta + \theta + \xi - \delta(\psi + \zeta) = 1 - \delta$;
(iii) $0 \le \psi + \zeta \le \min(1 + \varepsilon/\gamma, 1 + \zeta/\delta)$. *QED.*

While Assumption 1(i) assures that physical and human capital are both productive in each sector, Assumption 1(ii) requires socially constant returns to scale with respect to K and H in order to guarantee the existence of a BGP with a perpetual growth rate. Finally, in Assumption 1(iii), the restriction to $\psi + \zeta$, such that the aggregate congestion effect in Equation 3 is less than 100%, ensures the dominance of the positive effects over the congestion effects in the public services and thus avoids private increasing returns to scale.

The motions of the two kinds of capital stock are

$$\dot{k} = (1 - \tau_k)rvk + (1 - \tau_l)wuh + \pi - C - \varsigma k, \tag{4}$$

$$\dot{h} = X - \varsigma h, \tag{5}$$

where C denotes consumption, r and w are unit rewards to physical capital and to labor/human capital, respectively, π is the profits distributed to the household, and $\tau_k \in (-1, 1)$ and $\tau_l \in (-1, 1)$ are the tax/subsidy rates on capital and human capital, respectively. For simplicity, we assume the depreciation rates of physical capital and human capital are the same and are denoted as ς. Assuming different rates of depreciation for physical and human capital offers no extra insights, except for the cost of tedious algebra. The taxation scheme implicitly assumes that the education output is not a market good and thus is not subject to taxation, an assumption in line with Ben-Gad (2003).[4] Note that the flow budget and the non-market education output imply the definition of distributed profits: $\pi = Y - rvK - wuH$.

The representative agent is assumed to possess the following lifetime utility, with a constant, intertemporal elasticity of substitution consistent with a perpetual growth framework,

$$U = \int_0^\infty e^{-\rho t} \frac{C^{1-\sigma} - 1}{1 - \sigma} dt, \tag{6}$$

in which $\rho > 0$ is the instantaneous time-preference rate, and σ is the reciprocal of the intertemporal elasticity of substitution.

Finally, the government budget satisfies

$$G = T \equiv \tau_l wuH + \tau_k rvK = (\alpha\tau_k + \beta\tau_l)Y \equiv gY, \tag{7}$$

where $g \equiv \tau_k\alpha + \tau_l\beta$ denotes the share of government spending in output in the goods sector.

As the model stands, it agrees with Chen and Lee (2007) in that public good and its congestion appear. There are two major differences. First, we explicitly consider sector-specific externalities as arguments in the two production technologies in Equations 2 and 3 and distortionary factor taxes in the budget constraints in Equations 4 and 7 here, but they do not do so in Chen and Lee. Chen and Lee only modeled public goods in production technology and used income taxes in the budget constraint in the same way Barro (1990) did in a one-sector model. Second, and more importantly, we show that the argument in favor of indeterminacy based on sector-specific externalities and distortionary factor taxes is not robust once there are productive public goods. This point has never been made in Chen and Lee. As sector-specific externalities and distortionary factor taxes are both popular mechanisms in establishing indeterminacy, following Chen and Lee in the end we will introduce congestion in order to salvage these two mechanisms.

3. Optimization and Equilibrium

The necessary conditions of the household and the representative firm problems are standard and are delegated to Appendix A. Given tax rates τ_k and τ_l and initial capital $K(0)$ and $H(0)$, a perfect foresight market equilibrium is determined by production technologies, laws of motion, household optimization, firm optimization, and government budget balance.

In order to analyze the equilibrium, we transform the equilibrium conditions into a system with stationary variables. We accomplish this by following Bond, Wang, and Yip (1996) by denoting p as the shadow price of human capital, μ, in terms of physical capital, λ, and we transform the system into a structure with the following three variables, $\{p, s, z\}$, where $p \equiv \mu/$

[4] If education is a market activity and factor incomes from this sector are taxed, there are intersectoral externalities from the education sector to the commodity sector. Then, as the effect of intersectoral externalities on the goods sector offers an additional quantitative effect offsetting the effects of the sector-specific externalities and factor tax rates, the two existing mechanisms are even more fragile.

λ, $s \equiv C/H$, and $z \equiv K/H$.[5] In what follows we briefly explain the transformation, with detailed derivation found in Appendix A.

First, if we use the household's optimal conditions for v and u, together with r and w, we obtain a positive relationship between u and v, denoted as

$$v = v(u), \quad 0 < v < 1, 0 < u < 1,$$

and $v'(u) > 0$ because of the Pareto complements in K and H in the technology.

Next, we use the household's optimal conditions for u, v, K, and H, together with r and w, to obtain the following relationship:

$$u = u(p,z), \quad \frac{\delta u}{\delta p} > (\text{resp.} <) 0 \text{ and } \frac{\delta u}{\delta z} < (\text{resp.} >) 0 \quad \text{if} \quad \Gamma_1 < (\text{resp.} >) 0, \tag{8}$$

where

$$\begin{aligned} \Gamma_1 \equiv\ & \{\Lambda_1[\alpha\theta(1 - \tau_k) - \eta\beta(1 - \tau_l)]v(1 - u)\} \\ & + \alpha\theta(1 - \tau_k)\{\delta(1 - \gamma)(1 - \psi - \zeta) - (\gamma - \delta)[\psi(1 - v) + \zeta(1 - u)]\}, \end{aligned} \tag{9}$$

$$\Lambda_1 \equiv \alpha(\theta + \xi) - \eta(\beta + \varepsilon) + \psi[\delta(1 - \alpha) - \gamma(1 - \eta)] + \zeta[\gamma\eta - \alpha\delta]. \tag{10}$$

Moreover, in the Appendix we have used the optimal conditions for C and K and Equation 8 to obtain the growth rate of consumption and have transformed the equilibrium conditions into a stationary, three-variable system:

$$\frac{\dot{p}}{p} = \frac{\dot{\mu}}{\mu} - \frac{\dot{\lambda}}{\lambda} = (1 - \tau_k)r(p,z) - \frac{w(p,z)}{p}, \tag{11a}$$

$$\frac{\dot{s}}{s} = \frac{\dot{C}}{C} - \frac{\dot{H}}{H} = \frac{(1 - \tau_k)}{\sigma} r(p,z) - \frac{\rho + \zeta}{\sigma} - \frac{[1 - u(p,z)]}{\theta}\frac{w(p,z)}{p} + \zeta, \tag{11b}$$

$$\frac{\dot{z}}{z} = \frac{\dot{K}}{K} - \frac{\dot{H}}{H} = \frac{(1 - g)v[u(p,z)]}{\alpha} r(p,z) - \frac{s}{z} - \frac{[1 - u(p,z)]}{\theta}\frac{w(p,z)}{p}, \tag{11c}$$

where

$$r(p,z) = N\left[\frac{[1 - u(p,z)]^{\delta(1-\zeta)}}{pu(p,z)^{\delta}[1 - v(u(p,z))]^{\delta\psi}}\right]^{\frac{\beta+\varepsilon-\gamma\zeta}{\Lambda_1}} [v(u(p,z))]^{\frac{\gamma\psi(\theta+\xi-\delta\zeta)}{\Lambda_1}} [u(p,z)]^{\frac{\gamma\zeta(\theta+\xi-\delta\zeta)}{\Lambda_1}} \geq 0,$$

$$\frac{w(p,z)}{p} = \Pi p^{\frac{\eta(1-\gamma)+\delta(\alpha-\psi)}{\Lambda_1}} \left[\frac{u(p,z)[1 - v(u(p,z))]^{\psi}}{[1 - u(p,z)]^{(1-\zeta)}}\right]^{\frac{\delta(\alpha-\gamma\psi)}{\Lambda_1}} [v(u(p,z))]^{\frac{-\gamma\psi(\eta-\delta\psi)}{\Lambda_1}} [u(p,z)]^{\frac{-\gamma\zeta(\eta-\delta\psi)}{\Lambda_1}} \geq 0,$$

$$N = \alpha(Ag^{\gamma})^{\frac{1}{1-\gamma}} \Phi^{\frac{\beta+\varepsilon-\gamma\zeta}{\Lambda_1}},$$

[5] Notations λ and μ are, respectively, the co-state variable associated with K and H in the household's optimization problem.

$$\Pi = \theta B(Ag)^{\frac{\delta}{1-\gamma}} \left(\frac{\beta\eta(1-\tau_l)}{\alpha\theta(1-\tau_k)} \right)^{\eta} \Phi^{\frac{-\eta(1-\gamma)-\delta(\alpha-\psi)}{\Lambda_1}}.$$

Equations 11a–11c determine p, s, and z. After determining p, s, and z, variables u, v, Y/H, X/H, G/H, $\dot{K}/K$, $\dot{H}/H$, $\dot{C}/C$, $\dot{\lambda}/\lambda$, and $\dot{\mu}/\mu$ are determined by other equations.

In steady state, the equilibrium is a BGP, under which p, s, z, u, and v are constant, and the quantity variables grow at a constant rate. As a result, $\dot{p}/p = \dot{s}/s = \dot{z}/z = 0$ and $\dot{C}/C$, $\dot{K}/K$, $\dot{H}/H$, $\dot{Y}/Y$, $\dot{X}/X$, and $\dot{G}/G$, are constant and equal along a BGP. In the Appendix, we have shown that under a reasonable condition, there exists a unique BGP with a positive economic growth rate.

4. Reasons for Local Determinacy and Indeterminacy

Before we proceed to analyze the local indeterminacy of BGP in the next section, we explain the underlying reasons in this section. It is useful to show the reduced forms of social technology of both sectors. Denote $\hat{A} \equiv (Ag^{\gamma})^{1/(1-\gamma)}$ and $\hat{B} \equiv B(gA)^{\delta/(1-\gamma)}$. These are given by

$$Y = \hat{A}(v)^{\alpha/(1-\gamma)}(u)^{(\beta+\varepsilon)/1-\gamma}(K)^{\alpha/(1-\gamma)}(H)^{(\beta+\varepsilon)/(1-\gamma)}, \tag{12a}$$

$$X = \hat{B}(1-v)^{\eta}(1-u)^{\theta+\xi}(v)^{\delta\alpha/(1-\gamma)}(u)^{\delta(\beta+\varepsilon)/(1-\gamma)}(K)^{\eta+\delta\alpha/(1-\gamma)} (H)^{\theta+\xi+\delta(\beta+\varepsilon)/(1-\gamma)}, \tag{12b}$$

for the case without congestion in public goods, and by

$$Y = \hat{A}(v)^{\alpha/(1-\gamma)}(u)^{(\beta+\varepsilon)/(1-\gamma)}(K)^{(\alpha-\gamma\psi)/(1-\gamma)}(H)^{(\beta+\varepsilon-\gamma\zeta)/(1-\gamma)}, \tag{13a}$$

$$X = \hat{B}(1-v)^{\eta}(1-u)^{\theta+\xi}(v)^{\delta\alpha/(1-\gamma)}(u)^{\delta(\beta+\varepsilon)/(1-\gamma)}(K)^{\eta+\delta(\alpha-\psi)/(1-\gamma)} (H)^{\theta+\xi+\delta(\beta+\varepsilon-\zeta)/(1-\gamma)}, \tag{13b}$$

for the case with congestion in public goods.

Looking at the above equations, we may find the essence depends on the presence of intersectoral external effects. To understand this more clearly, we consider

$$Y = \hat{A}(vK)^{\alpha}(uH)^{\beta}\left(\overline{vK}\right)^{\varphi_1}\left(\overline{uH}\right)^{\varphi_2}\overline{[(1-v)K]}^{\varphi_3}\overline{[(1-u)H]}^{\varphi_4}\left(\bar{K}\right)^{\varphi_5}(\bar{H})^{\varphi_6}, \tag{14a}$$

$$X = \hat{B}[(1-v)K]^{\eta}[(1-u)H]^{\theta}\overline{[(1-v)K]}^{\lambda_1}\overline{[(1-u)H]}^{\lambda_2}\left(\overline{vK}\right)^{\lambda_3}\left(\overline{uH}\right)^{\lambda_4}(\bar{K})^{\lambda_5}(\bar{H})^{\lambda_6}, \tag{14b}$$

where $\alpha + \beta + \sum_{i=1}^{6}\phi_i = 1$ and $\eta + \theta + \sum_{i=1}^{6}\lambda_i = 1$ under constant returns.

In the above equations, each production sector can enjoy intersectoral as well as sector-specific external effects generated by physical and human capital and subject to congestion effects. In the absence of congestion effects ($\varphi_5 = \varphi_6 = \lambda_5 = \lambda_6 = 0$), Benhabib, Meng, and Nishimura (2000) and Mino (2001) assume that there are no intersectoral externalities in both sectors ($\varphi_3 = \varphi_4 = \lambda_3 = \lambda_4 = 0$). This paper implicitly assumes that education is not a market activity, so $\varphi_3 = \varphi_4 = 0$. As a result, intersectoral external effects only emerge from the commodity sector to the education sector. By letting $\varphi_1 = \gamma\alpha/(1 - \gamma)$ and $\varphi_2 = \gamma(\beta + \varepsilon)/(1 - \gamma)$ in Equation 14a, we obtain Equation 12a. If we assume $\lambda_1 = 0$, $\lambda_2 = \xi$, $\lambda_3 = \delta\alpha/(1 - \gamma)$, and $\lambda_4 = \delta(\beta + \varepsilon)/(1 - \gamma)$ in Equation 14b, we obtain Equation 12b. Introducing factor income taxation generates another quantitative effect. As productive public goods bring in intersectoral externalities that are an additional quantitative effect, the intersectoral externalities lead to determinacy unless the intersectoral externalities are relatively small (i.e., δ is small). When δ takes a small value, λ_3 and λ_4 in Equation 14b are small enough to make the model resemble that of Benhabib, Meng, and Nishimura (2000), in which there are no intersectoral externalities. In the presence of congestion effects, if we let $\varphi_5 = -\gamma\psi/(1 - \gamma)$, $\varphi_6 = -\gamma\zeta(1 - \gamma)$, $\lambda_5 = -\delta\psi(1 - \gamma)$, and $\lambda_6 = -\delta\zeta/(1 - \gamma)$, we obtain Equations 13a and 13b. Then, a larger congestion effect (a large ψ and ζ) plays exactly the same role as that of a smaller δ.

The reason for the above local indeterminacy result is as follows: Under constant returns to scale in a technology, the public service congestion reduces the contribution of public capital services and thus increases the contribution of physical and human capital from the social perspective, thereby mitigating the speed of diminishing marginal product. When the degrees of public service externalities and congestion effects are sufficiently large, the effect of diminishing return from the private perspective is dominated. Similarly, with Pareto complements in physical and human capital in the technology, the relative marginal product of human capital is also increasing in $uH/[(1 - u)H]$ from the social perspective if the degrees of public service externalities and congestion effects are sufficiently large.

Therefore, when the representative individual expects that the returns in Sector Y are relatively higher, he will allocate more resources from Sector X to Sector Y if the degrees of public good externality and congestion effects are sufficiently large, so the law of diminishing return does not hold from the social perspectives. Then, higher vK and uH in Sector Y make the private marginal returns of physical capital and human capital higher in Sector Y. As a consequence, expectations are self-fulfilling in equilibrium.

Specifically, we may analyze the relative rewards to both sectors as follows. As the relative marginal products of human capital and those of physical capital behave in a similar fashion, below we focus on the marginal products of human capital. In a competitive market, labor demand in a sector is determined by the equalization of the wage rate and the marginal product of human capital. Using Equations 1, 2, 13a, and 13b, the relative labor demand in Sector Y from the private perspective is

$$\frac{w_Y}{w_X} = \frac{MPH_y}{pMPH_x} = \frac{\beta}{\theta p}\frac{1-u}{u}\frac{Y}{X} = \Phi_1 \frac{1}{p}(vu)^{\frac{\zeta(\gamma-\delta)}{1-\gamma}}\left(\frac{vu}{(1-v)(1-u)}\right)^{\zeta\delta}\left(\frac{u}{1-u}\right)^{-\delta}\left[\left(\frac{v}{u}\right)z\right]^{\frac{\Lambda_1}{1-\gamma}}, \tag{15a}$$

where

$$\Phi_1 \equiv \frac{(1-\tau_l)\beta A^{1/(1-\gamma)} g^{\gamma/(1-\gamma)}}{\left[\theta B (gA)^{\delta/(1-\gamma)}\left(\frac{\beta\eta(1-\tau_l)}{\alpha\theta(1-\tau_k}\right)^{\eta}\right]}.$$

The relative demand for labor in Sector Y with respect to u is

$$\frac{\partial(w_Y/w_X)}{\partial u} = \frac{-w_Y}{w_X}\frac{\Gamma_1}{u(1-u)\alpha\theta}, \tag{15b}$$

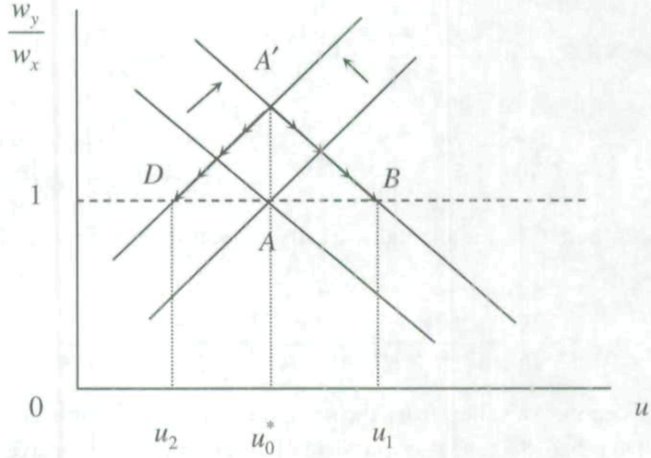

Figure 1. Relative Marginal Products of Labor

where Γ_1 is defined in Equation 9. The above relationship indicates that the sign of the relative labor demand curve in Sector Y is opposite to the sign of Γ_1. Therefore, the sign of Γ_1 is crucial.

When $\Gamma_1 > 0$, the relative labor demand curve is negative sloping in u (see Figure 1). With labor mobility, the wage rates and the marginal products of human capital must be equal at optimum, and thus $w_Y/w_X = 1$ in Equation 15a at optimum. Assume that the original optimum is at point A with $u = u_0^*$, and suppose that there is an increase in capital stock. Then z is increased to shift upward the relative labor demand curve w_Y/w_X in Equation 15a to point A′. At optimum, w_Y/w_X decreases along the negative sloping curve toward point B until $w_Y/w_X = 1$, and u is thus increased to u_1. Therefore, output Y increases.

Alternatively, when $\Gamma_1 < 0$, the relative labor demand curve is positive sloping in u. A larger capital stock increases z and thus shifts upward the relative labor demand curve to point A′. As the relative wage in Sector Y is larger than in Sector X, to equalize the wage the marginal product of labor/human capital in Sector X needs to increase relative to Sector Y. The fraction of labor employed in Sector Y is thus reduced along the positive sloping curve until point D, where the relative wage is returned to 1. As a result, output Y is reduced relative to Sector X under the case $\Gamma_1 < 0$.

The sign of Γ_1 may depend upon the factor intensity at the private perspective and at the social perspective. The physical capital intensity in Sector Y relative to Sector X at the private perspective is

$$\frac{vK/(uH)}{(1-v)K/[(1-u)H]} = \frac{v(1-u)}{u(1-v)} = \frac{\alpha\theta(1-\tau_k)}{\eta\beta(1-\tau_l)}, \tag{16}$$

where the household's and the firm's optimal conditions are used in the last equality. Denote $\Delta^P \equiv \alpha\theta(1-\tau_k) - \eta\beta(1-\tau_l)$. If $\Delta^P > (<)\ 0$, then Sector Y is physical (human) capital intensive from the private perspective.

Alternatively, the relative factor intensity at the social perspective is determined as follows: The optimal fractions of physical and human capital allocation between the two sectors, v^s and u^s, are derived in the social planner's problem taking into account the external effects. Using the technology from the social perspective in Equations 13a and 13b, the optimization conditions are

$$\lambda(1-g)\frac{\alpha}{1-\gamma}\frac{Y}{v^s} = \mu\left(\eta\frac{X}{1-v^s} - \frac{\delta\alpha}{1-\gamma}\frac{X}{v^s}\right), \tag{17a}$$

$$\lambda(1-g)\frac{\beta+\varepsilon}{1-\gamma}\frac{Y}{v^s} = \mu\left((\theta+\xi)\frac{X}{1-v^s} - \frac{\delta(\beta+\varepsilon)}{1-\gamma}\frac{X}{v^s}\right). \tag{17b}$$

Using Equations 17a and 17b, the capital intensity in Sector Y to Sector X from the social perspective is

$$\frac{v^s K^s/(u^s H^s)}{(1-v^s)K^s/[(1-u^s)H^s]} = \frac{v^s(1-u^s)}{u^s(1-v^s)} = \frac{\alpha(\theta+\xi)}{\eta(\beta+\varepsilon)}, \tag{18}$$

where superscript s denotes variables from the social perspectives. Denote $\Delta^s \equiv \alpha(\theta+\xi) - \eta(\beta+\varepsilon)$. If $\Delta^s > (<)\ 0$, then Sector Y is physical (human) capital intensive from the social perspective.

We are ready to determine the sign of Γ_1 in different set-ups in the model.

First, consider only the case with the distorted factor taxation and without sector-specific externalities and productive public spending (i.e., $\varepsilon = \xi = \gamma = \delta = 0$, $\tau_l > 0$ and $\tau_k > 0$). Then, we obtain

$$\Gamma_1^1 = \Delta^p \Delta^{s1} v(1-u), \tag{19a}$$

where $\Delta^{s1} \equiv \alpha\theta - \eta\beta$.

In order to obtain a positive sloping relative factor demand curve, it requires $\Gamma_1^1 < 0$ [i.e., $\text{sign}(\Delta^p) \neq \text{sign}(\Delta^{s1})$]. The condition is met when the gap is large in the two tax rates, τ_k and τ_l. This is found in Bond, Wang, and Yip (1996) and Raurich (2001).[6]

Second, consider only the sector-specific externalities without both the distorted factor taxation and the productive public spending (i.e., $\tau_l = \tau_k = \gamma = \delta = 0$, $\varepsilon > 0$, and $\xi > 0$). It requires $\text{sign}(\Delta^{p1}) \neq \text{sign}(\Delta^s)$ for $\Gamma_1^1 < 0$ in Equation 19a, where $\Delta^{p1} \equiv \alpha\theta - \eta\beta$. The condition is met when the difference between ξ and ε is large. The model is found in Benhabib, Meng, and Nisihmura (2000) and Mino (2001).

Third, consider both the sector-specific externalities and the distortionary factor taxation without productive public spending (i.e., $\varepsilon > 0$, $\xi > 0$, $\tau_k > 0$, $\tau_l > 0$, and $\gamma = \delta = 0$). It requires $\text{sign}(\Delta^p) \neq \text{sign}(\Delta^s)$ for $\Gamma_1^1 < 0$ in Equation 19a. The condition is met if the difference between $\tau_k - \tau_l$ and $\xi - \varepsilon$ is large. The model is found in Ben-Gad (2003).

A remark should be made for the above three cases in existing wisdom. To obtain $\Gamma_1 < 0$ in order for local indeterminacy, the existing works require the relative physical capital intensity in Sector Y to Sector X at the private perspective to be in reverse compared to that at the social perspective. The reversal breaks the law of diminishing marginal returns so the relative labor demand curve is positive sloping from the social perspective.

Fourth, consider the distortionary factor taxation and the sector-specific externalities with productive public spending and without congestion effect (i.e., $\psi = \zeta = 0$, $\varepsilon > 0$, $\xi > 0$, $\tau_k > 0$, $\tau_l > 0$, $\gamma > 0$, and $\delta > 0$). In this case, Γ_1 becomes

---

[6] The factor taxation in the educational sector is also considered in Bond, Wang, and Yip (1996) and Raurich (2001), and here we only consider the factor taxation on the goods sector. Nevertheless, the main features are the same.

$$\Gamma_1^2 = \Delta^p\Delta^s v(1-u) + \delta(1-\gamma)(1-\tau_k)\alpha\theta = \Delta^p\Delta^s v(1-u) + \delta(\alpha+\beta+\varepsilon)(1-\tau_k)\alpha\theta. \quad (19b)$$

If we compare Equation 19b with Equation 19a, Equation 19b involves a new term, $\delta(1-\gamma)(1-\tau_k)\alpha\theta > 0$, whose effect arises from productive public spending. As the productive public spending comes from taxing Sector Y, the new term captures an intersectoral external effect from Sector Y to Sector X.

When the fraction of human capital allocated to Sector Y increases, output Y, and thus public spending, increases. Through the intersectoral external effect, the marginal product of human capital in Sector X is increased, thereby reducing the possibility of a positive sloping relative labor demand curve. As a result, Γ_1^2 may be positive even if the difference between τ_k and τ_l, and the difference between ξ and ε are both large, causing these conventional mechanisms for local indeterminacy to stop working.

Finally, consider the congestion effect in use of public service, so $\psi > 0$ and $\zeta > 0$. Then, Γ_1 is the whole expression in Equation 9. We may rewrite it as

$$\begin{aligned}
\Gamma_1 &= \Delta^p\{\Delta^s - \psi[\gamma(1-\eta) - \delta(1-\alpha)] - \zeta[\alpha\delta - \gamma\eta]\}v(1-u) \\
&\quad + \delta(1-\gamma)(1-\tau_k)\alpha\theta(1-\psi-\zeta) - [\psi(1-v) + \zeta(1-u)](\gamma-\delta)\alpha\theta(1-\tau_k) \\
&= \Gamma_1^2 - \psi\{\Delta^p[\gamma(1-\eta) - \delta(1-\alpha)]v(1-u) + \delta(1-\gamma)(1-\tau_k)\alpha\theta \qquad (19c) \\
&\quad + (\gamma-\delta)(1-v)\alpha\theta(1-\tau_k)\} - \zeta\{(\alpha\delta - \gamma\eta)v(1-u) + \delta(1-\gamma)(1-\tau_k)\alpha\theta \\
&\quad + (\gamma-\delta)(1-u)\alpha\theta(1-\tau_k)\}.
\end{aligned}$$

There are three terms in the first equality in Equation 19c. Note that compared with Equation 19b, the first term is reduced by the congestion effect when $\gamma(1-\eta) > \delta(1-\alpha)$ and $\alpha\delta > \gamma\eta$. Moreover, the congestion effect makes the second term smaller by a fraction of $1-\psi-\zeta$, as compared with the second term in Equation 19b. Furthermore, the third term makes Γ_1 smaller if $\gamma > \delta$. Therefore, the congestion effect may lower Γ_1 so $\Gamma_1 < 0$. Intuitively, the congestion effect increases the marginal contribution of physical and human capital and thus mitigates the decrease in the marginal products of physical and human capital. When the congestion effect on Sector Y is larger than that on Sector X, the negative sloping relative labor demand curve rotates to become positive sloping. In particular, the result $\Gamma_1 < 0$ no longer requires $\text{sign}(\Delta^p) \neq \text{sign}(\Delta^s)$. Thus, local indeterminacy is possible even if the factor intensity at the social perspective is not in reverse from that at the private perspective, as long as the congestion effect is large enough.

5. Local Determinacy and Indeterminacy

The local dynamics of the economic system are analyzed by taking a linear Taylor expansion of Equations 12a–12c around the unique BGP. The linearization leads to the following:

$$\begin{pmatrix} \dot{p} \\ \dot{s} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} J_{11} & 0 & J_{13} \\ J_{21} & 0 & J_{23} \\ J_{31} & -1 & J_{33} \end{pmatrix} \begin{pmatrix} p - p^* \\ s - s^* \\ z - z^* \end{pmatrix}, \tag{20}$$

where

$$r(p,z) = N\left[\frac{[1 - u(p,z)]^{\delta(1-\zeta)}}{pu(p,z)^{\delta}[1 - v(u(p,z))]^{\delta\psi}}\right]^{\frac{\beta+\varepsilon-\gamma\zeta}{\Lambda_1}} [v(u(p,z))]^{\frac{\gamma\psi(\theta+\xi-\delta\zeta)}{\Lambda_1}} [u(p,z)]^{\frac{\gamma\zeta(\theta+\xi-\delta\zeta)}{\Lambda_1}} \geq 0,$$

$$\frac{w(p,z)}{p} = \Pi p^{\frac{\eta(1-\gamma)+\delta(\alpha-\psi)}{\Lambda_1}} \left[\frac{u(p,z)[1 - v(u(p,z))]^{\psi}}{[1 - u(p,z)]^{(1-\zeta)}}\right]^{\frac{\delta(\alpha-\gamma\psi)}{\Lambda_1}} [v(u(p,z))]^{\frac{-\gamma\psi(\eta-\delta\psi)}{\Lambda_1}} [u(p,z)]^{\frac{-\gamma\zeta(\eta-\delta\psi)}{\Lambda_1}} \geq 0,$$

$$N = \alpha(Ag^{\gamma})^{\frac{1}{1-\gamma}} \Phi^{\frac{\beta+\varepsilon-\gamma\zeta}{\Lambda_1}},$$

$$\Pi = \theta B(Ag)^{\frac{\delta}{1-\gamma}} \left(\frac{\beta\eta(1 - \tau_l)}{\alpha\theta(1 - \tau_k)}\right)^{\eta} \Phi^{\frac{-\eta(1-\gamma)-\delta(\alpha-\psi)}{\Lambda_1}},$$

where

$$J_{11} = \frac{1}{\Gamma_1}(\Sigma_1 - \Sigma_2)p^*,$$

$$J_{13} = \frac{\alpha\theta\Lambda_1}{\Gamma_1}(\Sigma_3 - \Sigma_4)p^* \frac{u^*(1 - u^*)}{z^*},$$

$$J_{21} = \left[\frac{1}{\sigma\Gamma_1}\Sigma_1 - \frac{1}{\sigma\Gamma_1}\Sigma_2(1 - u^*) - \frac{(1 - \tau_k)(1 - \gamma)\alpha}{\Gamma_1}\frac{r^* u^*(1 - u^*)}{p^*}\right]s^*,$$

$$J_{23} = \frac{\alpha\theta\Lambda_1}{\Gamma_1}\left[\frac{1}{\sigma}\Sigma_3 - \frac{1}{\theta}\Sigma_4(1 - u^*) + \frac{(1 - \tau_k)r^*}{\theta}\right]s^* \frac{u^*(1 - u^*)}{z^*},$$

$$J_{31} = \left\{\frac{(1 - g)}{\alpha(1 - \tau_k)\Gamma_1}\Sigma_1 v^* - \frac{1}{\theta\Gamma_1}\Sigma_2(1 - u^*) - \frac{(1 - \tau_k)(1 - \gamma)}{\Gamma_1}\right.$$
$$\left. \times [\alpha(1 - \tau_k)u^*(1 - u^*) + (1 - g)\theta v^*(1 - v^*)]\frac{r^*}{p^*}\right\}z^*,$$

$$J_{33} = \frac{\alpha\theta\Lambda_1}{\Gamma_1}\left\{\frac{(1 - g)}{\alpha(1 - \tau_k)}\left[\Sigma_3 v^* + \frac{v^*(1 - v^*)(1 - \tau_k)r^*}{u^*(1 - u^*)}\right] - \frac{1}{\theta}\Sigma_4(1 - u^*) + \frac{(1 - \tau_k)r^*}{\theta}\right\}$$
$$\times u^*(1 - u^*) + \frac{s^*}{z^*},$$

$$\Sigma_1 = \frac{-(1 - \tau_k)}{\Lambda_1}\{(\beta + \varepsilon - \gamma\zeta)[(1 - \psi v^* - \zeta u^*)\delta\alpha\theta(1 - \tau_k)(1 - \gamma) + \Gamma_1]$$
$$+ (\theta + \xi - \delta\psi)\alpha\theta(1 - \tau_k)(1 - \gamma)\gamma[\psi(1 - v^*) + \zeta(1 - u^*)]\}\frac{r^*}{p^*},$$

$$\Sigma_2 = \frac{(1 - \tau_k)}{\Lambda_1}\{\alpha\theta(1 - \tau_k)\delta(1 - \gamma)(\alpha - \gamma\psi)(1 - \psi v^* - \zeta u^*) + [\eta(1 - \gamma) + \delta(\alpha - \psi)]\Gamma_1$$
$$- (\eta - \delta\psi)\alpha\theta(1 - \tau_k)\gamma(1 - \gamma)[\psi(1 - v^*) + \zeta(1 - u^*)]\}\frac{r^*}{p^*},$$

$$\Sigma_3 = \frac{(1-\tau_k)^2}{\Lambda_1}\{\gamma(\theta+\xi-\delta\zeta)[\psi(1-v^*)+\zeta(1-u^*)]-\delta(\beta+\varepsilon-\gamma\zeta)[1-\psi v^*-\zeta u^*]\} \times \frac{r^*}{u^*(1-u^*)},$$

$$\Sigma_4 = \frac{(1-\tau_k)^2}{\Lambda_1}\{-\gamma(\eta-\delta\psi)[\psi(1-v^*)+\zeta(1-u^*)]+\delta(\alpha-\gamma\psi)(1-\psi v-\zeta u)\} \times \frac{r^*}{u^*(1-u^*)}.$$

In investigating the stability of Equation 20, there are two control-like variables, p and s, that can adjust immediately, and there is a state-like variable, z, the initial value of which is predetermined and can adjust only gradually. Let ω_1, ω_2, and ω_3 be the three eigenvalues. Then, they satisfy

$$\begin{aligned}\omega_1+\omega_2+\omega_3 &= TrJ = J_{11}+J_{33},\\ \omega_1\omega_2+\omega_1\omega_3+\omega_2\omega_3 &= BJ = J_{11}J_{33}-J_{13}J_{31}+J_{23},\\ \omega_1\omega_2\omega_3 &= DetJ = J_{11}J_{23}-J_{21}J_{13},\end{aligned}$$

in which TrJ is the trace, BJ is the determinant of the Bordered Hessian, and $DetJ$ is the determinant of the Jacobean matrix in Equation 20, with the Jacobean matrix denoted as J.

Given any initial value of the state-like variable, the equilibrium path in the neighborhood of the unique BGP is locally determinate if J has exactly one eigenvalue with negative real parts and is locally indeterminate if J has the number of eigenvalues with negative real parts larger than or equal to 2. In applying the Routh-Hurwitz theorem, the number of characteristic roots with positive real parts equals the number of variations of signs in $\{-1, TrJ, -BJ+DetJ/TrJ, DetJ\}$. There are a total of eight possible types of variations in sign. The determinant is

$$DetJ = \frac{-(1-\tau_k)^3\alpha\theta u^*(1-u^*)s^*r^{*2}}{z^*\Gamma_1} \times \left\{\left(\frac{1}{\sigma}-\frac{1-u^*}{\theta}\right)\frac{\delta(\beta+\varepsilon)}{u^*(1-u^*)}+\frac{(\beta+\varepsilon)(1+\eta)+\alpha(\eta+\delta)}{\theta}\right\}, \tag{21}$$

whose sign is opposite to the sign of Γ_1. Except for the case with three eigenvalues having negative real parts, local indeterminacy arises only when there are two eigenvalues with negative real parts. When this happens, $DetJ > 0$.

In Appendix B, we have shown that under mild conditions, the determinant of the Bordered Hessian is negative. However, the determinant of the Bordered Hessian is the sum of the products of any two eigenvalues. If all of the three eigenvalues have negative real parts, the determinant of the Bordered Hessian must be positive rather than negative. Therefore, local indeterminacy is possible only under the case with only two eigenvalues having negative real parts. As a result, local indeterminacy arises only under $DetJ > 0$, which requires $\Gamma_1 < 0$. To summarize, we find:

PROPOSITION 1. Under Assumption 1 and Condition S, local indeterminacy is possible only under $\Gamma_1 < 0$. *QED*.

Table 1.

A	Growth Rate	τ_l	g	γ	δ	ε	ξ	ς
1	0.02	0.31	0.2	0.08	0.08	0.05	0.05	0.05

If there are no public goods in this model (i.e., $\gamma = \delta = 0$), it is known that the possibility of indeterminacy arises when social and private factor intensities differ, as they did in Bond, Wang, and Yip (1996), Benhabib, Meng, and Nishimura (2000), Mino (2001), Raurich (2001), and Ben-Gad (2003). To see this point specifically, when $\gamma = \delta = 0$, Γ_1 becomes

$$\Gamma_1 = \Delta^s \Delta^p v(1-u) \equiv [\alpha(\theta + \xi) - \eta(\beta + \varepsilon)][\alpha\theta(1-\tau_k) - \eta\beta(1-\tau_l)]v(1-u).$$

The condition for indeterminacy is the different signs of Δ^s and Δ^p (i.e., different factor intensity between the social perspective and the private perspective). It is easy to see that this condition is met under reasonable parameters space that will be used below.[7]

If there are productive public goods, whether $\Gamma_1 < 0$ is a quantitative issue. Our quantitative method below reveals that even under a large gap in factor income tax rates and in sector-specific externalities it is impossible to obtain $\Gamma_1 < 0$, unless there are congestion effects.

Public Goods without Congestion Effects

When public goods exhibit no congestion effects, $\psi = \zeta = 0$. As we will find below, the economy no longer exhibits local indeterminacy because the effect of intersectoral externalities on the production in the education sector is so large that it offsets the effects of both factor taxation and sector-specific externalities that would have otherwise reversed the factor intensity at a social level from that at a private level. In particular, with the sources of the public spending coming from the goods sector, public spending increases the productivity in the education sector via an intersectoral external effect, which under a social constant return dampens the marginal returns in this sector and makes the relative factor demand curve impossible to slope positively. As a consequence, the two conventional mechanisms are not robust in the establishment of local indeterminacy in models with productive public goods. In what follows, we demonstrate that local indeterminacy is impossible quantitatively.

We start by calibrating the model economy. We choose parameter values, followed by solving the endogenous variables in a BGP. We normalize the parameter values of the productivity coefficient in Sector Y by $A = 1$. We calibrate the economy based upon the following parameter values representative of the U.S. economy and consistent with a 2% long-run, real economic growth rate. We summarize these parameters in Tables 1 and 2.

We choose $\varepsilon = \xi = 0.05$, following the work of Benhabib and Nishimura (1998). Moreover, we set $\tau_l = 0.31$, in line with Jones, Manuelli, and Rossi (1993). The depreciation rates of both capitals are chosen as $\varsigma = 0.05$, in accordance with Mulligan and Sala-i-Martin (1993) and Raurich (2001). Public expenditure as a fraction of income is set at $g = 0.2$, consistent with the average size of the government in the United States. Next, for government

[7] For example, for the set of values $\{\alpha = 0.3, \eta = 0.27, \varepsilon = 0, \xi = 0.02, \tau_l = 0.17, \tau_k = 0.27\}$, with the value of τ_k calibrated in accord with the average size of the government in the United States at $g = 0.2$, Δ^s and Δ^p have different signs.

Table 2.

Cases	ρ	σ	α	η	τ_k	B
I	0.05	1.5	0.285	0.2	0.0654	0.15298
II	0.025	1.5	0.25	0.2	0.0312	0.09156
III	0.04	1.1	0.35	0.2	0.1108	0.73475

externalities, some consider it to be as large as 30% (e.g., Aschauer 1989), and some consider it very small (e.g., Barro and Sala-i-Martin 1995). Following Turnovsky (2000), we choose the degree of externality of public spending upon the goods sector at $\gamma = 0.08$; in a symmetric fashion, we set the degree of externality of public spending upon the education sector at $\delta = 0.08$. This degree of government externalities is between these two extremes. Moreover, as there are congestion effects, the degree of government externalities is actually less than 0.08.

For the time preference rate, ρ, the intertemporal elasticity of substitution, $1/\sigma$, and the share of physical capital in the goods and the education sector, α and η, respectively, we choose three cases following Ben-Gad (2003), Benhabib and Perli (1994), and Raurich (2001), and we denote them as Cases I, II, and III, respectively, in Table 2.[8] We use the three sets of parameter values in order to assure the robustness of our quantitative results. With the parameter values in Table 1 and under constant returns to scale technology, the shares of human capital in the goods and the education sector, β and θ, are implied. The values of τ_k and B are then calibrated in accord with the U.S. economy, as shown in Table 2. Based on the above parameter values, we then simulate the model to assess the possibility of local indeterminacy.

Our simulation results indicate that local indeterminacy is implausible under a wide range of parameter values.[9] The reason for the result is that the public spending effect always quantitatively dominates the effects arising from both the sector-specific externalities and the factor income taxation, which are captured by the product of Δ^s and Δ^p in Equation 19b. This situation may change only under both a very small δ, as then the intersectoral externalities are very small and the model resembles that of Bond, Wang, and Yip (1996) and Benhabib, Meng, and Nishimura (2000). Yet a small δ implies a requirement of a very large τ_k, in accord with the model. In the Case I parameter values, for example, when $\delta = 0.02$, local indeterminacy emerges only when τ_k is as large as 0.7, with the required value of τ_k for local indeterminacy increasing in δ. However, the requirement for such a large average capital income tax rate seems too restrictive to be plausible.

Congestion Effects

With congestion effects, $\psi > 0$ and $\zeta > 0$. In this case, we will find that Γ_1 is lower when the congestion effect is large enough, and that this effect on Sector Y is greater than that on Sector X (i.e., $\psi > 0$, $\zeta > 0$, and $\gamma > \delta$).

[8] In Benhabib and Perli (1994), $\sigma = 1$ is due to the technical restriction for the existence of a BGP, and there is no value for η, as the education sector does not use physical capital. Here we do not require the restriction $\sigma = 1$, but rather we follow the method of Ben-Gad (2003) to choose $\sigma = 1.5 > 1$ and $\eta = .2$, the former of which has been chosen by Jones, Manuelli, and Rossi (1993) and the latter of which has been chosen by Mulligan and Sala-i-Martin (1993). Finally, we use the parameter values of the centralized economy in Raurich (2001) in Case III.

[9] As our simulation results indicate, under a large range of the externality of public spending [i.e., $\gamma \in (0.01, 0.4)$, $\delta \in (0.02, 0.4)$], the unique BGP is always locally determinate when $\tau_l \in (-0.7, 0.7)$, $\tau_k \in (-0.7, 0.7)$, $\varepsilon \in (0.01, 0.2)$, $\xi \in (0.01, 0.2)$, and $g \in (0.15, 0.35)$.

Table 3.

ρ	σ	α	η	ψ	ζ	τ_k	B
0.025	1.5	0.3	0.2	0.5	0.5	−0.005	0.09325

To assess the effects, parameter values for τ_l, g, γ, δ, ε, and ξ in Table 1 are used in accord with the U.S. economy. For other parameters, as we obtain similar results for the three cases in Table 2, to save space we only use one set of parameter values. We choose $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, and $\eta = 0.2$. Finally, for the degree of congestion, in a paper concerning fire protection services McMillan (1989) estimated it at the range of 0.499 and 0.921. We choose $\psi = \zeta = 0.5$ for the degree of congestion for both capitals. We then calibrate the model and obtain $\tau_k = 0.03613$ and $B = 0.08719$. The parameters are summarized in Table 3. We must point out that the resulting calibrated values for τ_k and B are insensitive to a different value of ψ and ζ.

We then simulate the model and envisage whether the BGP is a source, a sink, or a saddle. In the simulation, all the parameter values in Tables 1 and 3 are fixed, with the exception that we vary the values for the congestion (ψ and ζ), the public goods externalities (γ, δ), the sector-specific externalities (ε, ξ), and the factor tax rates (τ_l, τ_k).

The simulation results are that local indeterminacy emerges easily. For example, under $\varepsilon = 0$, $\xi = 0.02$, and $\tau_l = 0.2$ and the resulting $\tau_k \in (0.2, 0.4)$ in the requirement of government budget balance, we find that for the reasonable range of $\delta < 0.3$ and $\gamma < 0.3$, the BGP that is indeterminate emerges for all ψ and ζ in [0.11, 0.54]. We must point out that the parameter values on sector-specific externalities and factor tax rates are within a reasonable range. In particular, under these values there is no reversal in factor intensity at a private level and at a social level if $\psi = \zeta = 0$, which is verified by the positive value of the first term in the first equality in Equation 19c. Therefore, the congestion effect is a mechanism to regain local indeterminacy in a general two-sector Lucas growth model with productive public spending.

Now, we envisage how differences in the factor tax rates and in the sector-specific externalities affect the possibility of local indeterminacy. Note that it is necessary that $\Lambda_1 < 0$ in order for the first term in Equation 19c to be negative. Then, when $\tau_k - \tau_l > 0$ is larger, $[\alpha\theta(1 - \tau_k) - \eta\beta(1 - \tau_l)]$ is larger, enlarging the negative first term in Equation 19c. Alternatively, when $\xi - \varepsilon$ is larger, Assumption 1(ii) implies larger $\theta - \beta$ and thus larger $[\alpha\theta(1 - \tau_k) - \eta\beta(1 - \tau_l)]$, thereby enlarging the negative first term in Equation 19c. Therefore, a larger gap in $\tau_k - \tau_l$ and in $\xi - \varepsilon$ increases the possibility of local indeterminacy in models with congestible public services with a given congestion effect.

To quantify how a larger gap in $\tau_k - \tau_l$ and in $\xi - \varepsilon$ increases the possibility of local indeterminacy, we illustrate the effects under a given public goods congestion scenario. Specifically, we present the results under two sets of values: $\psi = \zeta = 0.2$ and $\psi = \zeta = 0.5$. The quantitative results are the same if different values of ψ and ζ are used. In Figures 2 and 3 below, we choose $\varepsilon = 0$ and the value of τ_k consistent with the government budget constraint and then simulate to obtain the set of (δ, γ) for a sink under three different sets for (ξ, τ_l) (i.e., (ξ, τ_l) = (0.02, 0.2), (0.04, 0.2), and (0.02, 0.15), respectively).

First we illustrate the effects in the case of $\psi = \zeta = 0.2$ in Figure 2. In the figure both the dark- and gray-shaded areas are the resulting set of (δ, γ) that exhibits indeterminate local dynamic paths in the neighborhood of the unique BGP. The figure is explained as follows: First, under (ξ, τ_l) = (0.02, 0.2) and the resulting $\tau_k \in (0.26, 0.33)$ in accord with the government budget balance, the dark-shaded area above the dotted line is the range of (δ, γ) that yields a

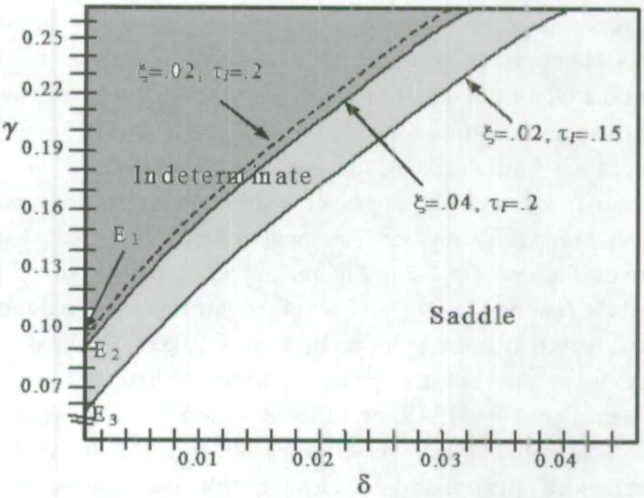

Figure 2. $\psi = \zeta = 0.2$. Notes: 1. $A = 1$, $B = 0.09325$, $g = 0.2$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$, $\varepsilon = 0$, $\varsigma = 0.05$, and $\psi = \zeta = 0.2$. 2. The dark-shaded area above the dotted line is the range for local indeterminacy under $\xi = 0.02$, $\tau_l = 0.2$, and $\tau_k \in (0.26, 0.32)$ with E_1 at $(\delta, \gamma) = (0, 0.098)$; the entire dark-shaded area is the region for indeterminacy under $\xi = 0.04$, $\tau_l = 0.2$, and $\tau_k \in (0.26, 0.32)$ with E_2 at $(0, 0.096)$; and the entire dark- and gray-shaded area is the region for indeterminacy under $\xi = 0.02$, $\tau_l = 0.15$, and $\tau_k \in (0.35, 0.40)$ with E_3 at $(0, 0.060)$.

BGP that is a sink, with the lowest point at E_1, $(\delta, \gamma) = (0, 0.098)$. The range in (δ, γ) sounds reasonable. Next, when $(\xi, \tau_l) = (0.04, 0.2)$ and the resulting $\tau_k \in (0.26, 0.32)$—when the sector-specific externality in the education sector is increased from 0.02 to 0.04—the range of (δ, γ) for local indeterminacy is expanded rightward to include the entire dark-shaded area, with the lowest point at E_2, $(0, 0.096)$, lower than E_1. Finally, when $(\xi, \tau_l) = (0.02, 0.15)$ and the resulting $\tau_k \in (0.35, 0.4)$—when the labor tax rate is reduced, thus enlarging the difference in factor tax rates—the region for local indeterminacy is expanded rightward to include the dark-

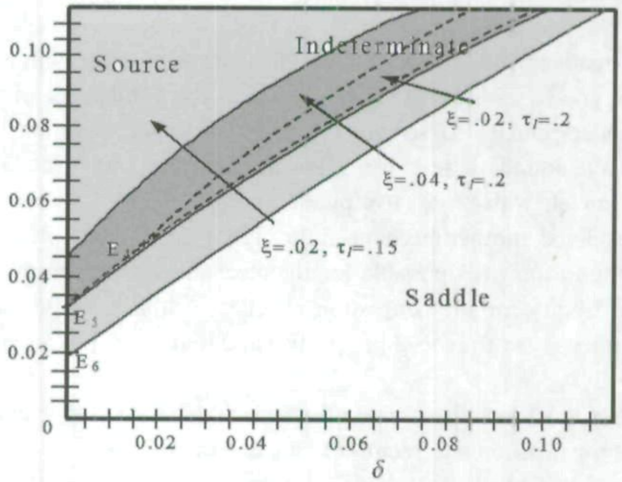

Figure 3. $\psi = \zeta = 0.5$. Notes: 1. $A = 1$, $B = 0.09325$, $g = 0.2$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$, $\varepsilon = 0$, $\varsigma = 0.05$, and $\psi = \zeta = 0.5$. 2. The area surrounded by the two dotted lines is the region for indeterminacy under $\xi = 0.02$, $\tau_l = 0.2$, and $\tau_k = 0.2$ with E_4 at $(\delta, \gamma) = (0.015, 0.052)$; the dark-shaded area is the region for indeterminacy under $\xi = 0.04$, $\tau_l = 0.2$, and $\tau_k = 0.2$ with E_5 at $(0, 0.033)$; and the entire dark- and gray-shaded area is the region for indeterminacy under $\xi = 0.02$, $\tau_l = 0.15$, and $\tau_k = 0.316$ with E_6 at $(0, 0.021)$.

and gray-shaded areas as a whole, with the lowest point at E_3, (0, 0.060), even lower than E_2. Thus, other things being equal, when there is a larger difference in the sector-specific externalities and in the factor tax rate, it is easier to yield local indeterminacy.

Next we illustrate the simulation results in the case in which $\psi = \zeta = 0.5$ in Figure 3. Two points are observed. First, both the dark- and gray-shaded areas are in the range of (δ, γ) that exhibits local indeterminacy, with the range for a source lying above it and the range for a saddle lying below it. Secondly, a smaller γ, as small as 0.021, may create local indeterminacy, as compared to the case of $\psi = \zeta = 0.2$ in Figure 1, in which the smallest γ is 0.060. Under $(\xi, \tau_l) = (0.02, 0.2)$ and the resulting $\tau_k = 0.2$, the set of (δ, γ) within the two dotted lines establishes local indeterminacy, with the lowest point at E_4, (0.015, 0.052). This is one of the cases with differences in sector-specific externality and without differences in factor tax rates. Alternatively, under $(\xi, \tau_l) = (0.04, 0.2)$ and the resulting $\tau_k = 0.2$—when the sector-specific externality in the education sector is increased from 0.02 to 0.04—the range of (δ, γ) for local indeterminacy is expanded leftward and includes the entire dark-shaded area, with the lowest point at E_5, (0, 0.033), lower than E_4. Finally, under $(\xi, \tau_l) = (0.02, 0.15)$ and the resulting $\tau_k = 0.316$—when labor income tax rate is reduced from 20% to 15%, and thus the gap between factor tax rates is increased—the dark- and gray-shaded areas as a whole generate local indeterminacy, with the lowest point at E_6, (0, 0.021), lower than E_5.

Finally, a remark should be made concerning a comparison between Figures 2 and 3. The difference between γ and δ required for local indeterminacy in Figure 3 is smaller than that in Figure 2. The result indicates that the congestion in public goods relaxes the required gap between γ and δ in the creation of local indeterminacy. In particular, we find that even if there is no reversal in factor intensity at the social perspective compared to that at the private perspective, the congestion effect can create indeterminacy.

6. Concluding Remarks

This paper extends existing two-sector, human capital–based growth models and considers productive public goods services. We have revisited the robustness of the mechanisms to establish local indeterminacy based on differential factor taxation and sector-specific externality. We have found these existing mechanisms are fragile in the creation of local indeterminacy in models with productive public goods.

We have considered another mechanism through the congestion effect in the use of public goods and have found that it is possible for the mechanism to generate local indeterminacy. Moreover, the possibility for the congestion effect to establish local indeterminacy emerges even if the factor intensity at the social perspective and that at the private perspective are not in reverse.

Finally, we have focused on searching for a mechanism on the technology side to salvage the differential factor taxation and sector-specific externalities. An extension in the preference side may work as well. For example, if we consider leisure following the methods of Palivos, Yip, and Zhang (2003), Raurich (2003), and Mino (2004), we may establish local indeterminacy. This is an avenue for further research.

Appendix A: Derivation of the Model

Given the tax rates, τ_l and τ_k, factor rewards, r and w, externalities, $\overline{(1-u)H}$, public spending, G, and initial $K(0)$ and $H(0)$, the representative agent chooses C, v, u, K, and H in order to maximize its discounted lifetime utility in Equation 6, subject to its constraints in Equations 1 and 2 and Equations 4 and 5. The Hamiltonian is

$$\Re = \frac{C^{1-\sigma}-1}{1-\sigma} + \lambda[(1-\tau_k)rvk + (1-\tau_l)wuh + \pi - C - \varsigma k] + \mu\left\{B[(1-v)k]^{\eta}[(1-u)h]^{\theta}\left[\overline{(1-u)H}\right]^{\xi}G^{\delta} - \varsigma h\right\},$$

where λ and μ are the co-state variables associated with k and h, respectively.

The necessary conditions, together $k = K$ and $h = H$ in equilibrium, are

$$C^{-\sigma} = \lambda, \tag{A1a}$$

$$\lambda(1-\tau_k)rK = \mu\eta\frac{X}{1-v}, \tag{A1b}$$

$$\lambda(1-\tau_l)wH = \mu\theta\frac{X}{1-u}, \tag{A1c}$$

$$\lambda[(1-\tau_k)rv - \varsigma] + \mu\eta\frac{X}{K} = \rho\lambda - \dot{\lambda}, \tag{A1d}$$

$$\lambda[(1-\tau_l)wu - \varsigma] + \mu\theta\frac{X}{H} = \rho\mu - \dot{\mu}, \tag{A1e}$$

along with the two transversality conditions, $\lim_{t\to\infty} e^{-\rho t}\lambda K = 0$ and $\lim_{t\to\infty} e^{-\rho t}\mu H = 0$.

The representative firm's profit-maximizing behavior is trivial. Assuming the competitive goods market, the optimal conditions are

$$r = \alpha Y/(vK) \quad \text{and} \quad w = \beta Y/(uH). \tag{A2}$$

First, if we divide Equation A1b by Equation A1c, together with r and w in Equation A2, we obtain the following:

$$v = \frac{(1-\tau_k)\alpha\theta u}{(1-\tau_k)\alpha\theta u + (1-\tau_l)\eta\beta(1-u)} \equiv v(u) < 1 \quad \text{if } u < 1,\ v'(u) > 0, \tag{A3}$$

indicating a positive relationship between u and v.

Denote $z = K/H$ and $p \equiv \mu/\lambda$. Substituting Equation A1b into Equation A1d, and substituting Equation A1c into Equation A1e, together with Equations A2, 6a, and 6b, we obtain

$$(1-\tau_k)\alpha\frac{Y}{vK} = (1-\tau_k)\alpha A^{\frac{1}{1-\gamma}}g^{\frac{1}{1-\gamma}}v^{\frac{\gamma\psi}{1-\gamma}}u^{\frac{\gamma\zeta}{1-\gamma}}\left(\frac{v}{u}z\right)^{\frac{-(\beta+\varepsilon-\gamma\zeta)}{1-\gamma}} \equiv (1-\tau_k)r(p,z) = \rho + \varsigma - \frac{\dot{\lambda}}{\lambda}, \tag{A4}$$

$$\begin{aligned}\theta\frac{X}{(1-u)H} &= \theta B(gA)^{\frac{\delta}{1-\gamma}}\left(\frac{\beta\eta(1-\tau_l)}{\alpha\theta(1-\tau_k)}\right)^{\eta}\left(\frac{u}{1-u}\right)^{\delta}v^{\frac{\delta\gamma\psi}{1-\gamma}}u^{\frac{\delta\gamma\zeta}{1-\gamma}}(1-v)^{\delta\psi}(1-u)^{\delta\zeta}\left(\frac{v}{u}z\right)^{\frac{\eta(1-\gamma)+\delta(\alpha-\psi)}{1-\gamma}} \\ &\equiv \frac{w(p,z)}{p} = \rho + \varsigma - \frac{\dot{\mu}}{\mu},\end{aligned} \tag{A5}$$

where

$$r(p,z) \equiv \alpha A^{\frac{1}{1-\gamma}}g^{\frac{\gamma}{1-\gamma}}v^{\frac{\gamma\psi}{1-\gamma}}u^{\frac{\gamma\zeta}{1-\gamma}}\left(\frac{v}{u}z\right)^{\frac{-(\beta+\varepsilon-\gamma\zeta)}{1-\gamma}},$$

$$\frac{w(p,z)}{p} \equiv \theta B(gA)^{\frac{\delta}{1-\gamma}}\left(\frac{\beta\eta(1-\tau_l)}{\alpha\theta(1-\tau_k)}\right)^{\eta}\left(\frac{u}{1-u}\right)^{\delta}v^{\frac{\delta\gamma\psi}{1-\gamma}}u^{\frac{\delta\gamma\zeta}{1-\gamma}}(1-v)^{\delta\psi}(1-u)^{\delta\zeta}\left(\frac{v}{u}z\right)^{\frac{\eta(1-\gamma)+\delta(\alpha-\psi)}{1-\gamma}}.$$

Next, with the help of Equations A4 and A5, we rearrange Equation A1b to obtain

$$\frac{(1-\tau_l)\beta(Ag^{\gamma}g)^{\frac{1}{1-\gamma}}}{\theta B(gA)^{\frac{\delta}{1-\gamma}}[\beta\eta(1-\tau_l)/(\alpha\theta(1-\tau_k))]^{\eta}}\left(\frac{u}{1-u}\right)^{-\delta}[v(u)]^{\frac{\psi\gamma(1-\delta)}{1-\gamma}}u^{\frac{\zeta\gamma(1-\delta)}{1-\gamma}}[(1-v(u))]^{-\delta\psi}[(1-u)]^{-\delta\zeta}\left(\frac{v(u)}{u}\right)^{\frac{\Lambda_1}{1-\gamma}} = z^{-\frac{\Lambda_1}{1-\gamma}}p. \tag{A6}$$

The above equation indicates relationship $u = u(p, z)$ in Equation 8, in which

$$\frac{\partial u}{\partial p} = \frac{-(1-\gamma)(1-\tau_k)\alpha\theta u(1-u)}{p\Gamma_1} \gtrless 0 \quad \text{if} \quad \Gamma_1 \gtrless 0,$$

$$\frac{\partial u}{\partial z} = \frac{\Lambda_1(1-\tau_k)\alpha\theta u(1-u)}{z\Gamma_1} \gtrless 0 \quad \text{if} \quad \Gamma_1 \gtrless 0,$$

$$\begin{aligned}\Gamma_1 \equiv {} & \{\Lambda_1[\alpha\theta(1-\tau_k) - \eta\beta(1-\tau_l)]v(1-u)\} \\ & + \alpha\theta(1-\tau_k)\{\delta(1-\gamma)(1-\psi-\zeta) - (\gamma-\delta)[\psi(1-v) + \zeta(1-u)]\},\end{aligned} \tag{9}$$

$$\begin{aligned}\Lambda_1 & \equiv \alpha(\theta+\xi) - \eta(\beta+\varepsilon) + \psi[\delta(1-\alpha) - \gamma(1-\eta)] + \zeta(\gamma\eta - \alpha\delta) \\ & \equiv \alpha(1-\delta) - \eta(1-\gamma) - \psi(\gamma-\delta).\end{aligned} \tag{10}$$

Equation A6 also implies that vK/uH is a function of u and p, and, therefore, the vK/uH in Equations A4 and A5 can be replaced by this relation.

Now, we transform the equilibrium into a system with three variables, $\{p, s, \text{and } z\}$. First, if we combine Equations A1a and A4, we get

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}[(1-\tau_k)r - \rho - \varsigma]. \tag{A7}$$

Next, denote $s \equiv C/H$ and $z \equiv K/H$. If we use Equations A4, A5, 3, and 4, we obtain

$$\frac{\dot{K}}{K} = \frac{(1-g)v}{\alpha}r - \frac{s}{z} - \varsigma, \tag{A8}$$

$$\frac{\dot{H}}{H} = \frac{(1-u)}{\theta}\frac{w}{p} - \varsigma, \tag{A9}$$

where

$$r(p,z) = N\left[\frac{[1-u(p,z)]^{\delta(1-\zeta)}}{pu(p,z)^{\delta}[1-v(u(p,z))]^{\delta\psi}}\right]^{\frac{\beta+\varepsilon-\gamma\zeta}{\Lambda_1}}[v(u(p,z))]^{\frac{\gamma\psi(\theta+\xi-\delta\zeta)}{\Lambda_1}}[u(p,z)]^{\frac{\gamma\zeta(\theta+\xi-\delta\zeta)}{\Lambda_1}} \geq 0,$$

$$\frac{w(p,z)}{p} = \Pi p^{\frac{\eta(1-\gamma)+\delta(\alpha-\psi)}{\Lambda_1}}\left[\frac{u(p,z)[1-v(u(p,z))]^{\psi}}{[1-u(p,z)]^{(1-\zeta)}}\right]^{\frac{\delta(\alpha-\gamma\psi)}{\Lambda_1}}[v(u(p,z))]^{\frac{-\gamma\psi(\eta-\delta\psi)}{\Lambda_1}}[u(p,z)]^{\frac{-\gamma\zeta(\eta-\delta\psi)}{\Lambda_1}} \geq 0,$$

$$N = \alpha(Ag^{\gamma})^{\frac{1}{1-\gamma}}\Phi^{\frac{\beta+\varepsilon-\gamma\zeta}{\Lambda_1}},$$

$$\Pi = \theta B(Ag)^{\frac{\delta}{1-\gamma}}\left(\frac{\beta\eta(1-\tau_l)}{\alpha\theta(1-\tau_k)}\right)^{\eta}\Phi^{\frac{-\eta(1-\gamma)-\delta(\alpha-\psi)}{\Lambda_1}}.$$

Finally, Equations 12a–12c are obtained by using Equations A4, A5, A7, A8, and A9.

Appendix B: Existence and Uniqueness of BGP and Positive Economic Growth Rate

To determine a BGP, we substitute Equation 11a into Equation 11b to obtain

$$\chi\left\{\frac{u}{(1-u)^{1-\zeta}}[1-v(u)]^{\psi}\right\}^{\frac{(\beta+\varepsilon-\gamma\zeta)\delta}{\Lambda_2}}\left[v(u)^{\psi}u^{\zeta}\right]^{\frac{\gamma[\eta+\delta(1-\psi)]}{\Lambda_2}}\left[\frac{1}{\sigma}-\frac{(1-u)}{\theta}\right]=\frac{\rho+(1-\sigma)\varsigma}{\sigma}, \tag{A10}$$

where

$$\chi=(1-\tau_k)\alpha A^{\frac{1}{1-\gamma}}g^{\frac{\gamma}{1-\gamma}}\left[\frac{(1-\tau_k)\alpha A^{\frac{1}{1-\gamma}}g^{\frac{\gamma}{1-\gamma}}}{\theta B(gA)^{\delta/(1-\gamma)}\{\beta\eta(1-\tau_l)/[\alpha\theta(1-\tau_k)]\}^{\eta}}\right]^{\frac{-(\beta+\varepsilon-\gamma\zeta)}{\Lambda_2}},$$

$$\Lambda_2=(1+\eta)(1-\gamma)-\alpha(1-\delta)+\psi(\gamma-\delta)>0.$$

Examining Equation A10, while the right-hand side is a constant and positive when $0<\rho+(1-\sigma)\varsigma/\sigma<\infty$, the left-hand side is a function of u, whose value is 0 at $u=1-\theta/\sigma$ and is positive for $u>1-\theta/\sigma$ under[10]

CONDITION S: $\sigma>\frac{1}{2}\left(\alpha(1-\tau_k)+\theta+\sqrt{[\alpha(1-\tau_k)+\theta]^2+4\left(\frac{\eta\beta(1-\tau_l)}{1-(\alpha\tau_k+\beta\tau_l)}-\alpha\theta(1-\tau_k)\right)}\right)$. *QED*

Moreover, the left-hand side is increasing in u for $u\geq 1-\theta/\sigma$ and is approaching ∞ at $u=1$. Therefore, there exists a unique interior u^*, $0<1-\theta/\sigma<u^*<1$, in a BGP, whose relationship $v=v(u)$ implies a unique interior v^*,

$$0<\frac{\alpha(1-\tau_k)(\sigma-\theta)}{\alpha(1-\tau_k)(\sigma-\theta)+\eta\beta(1-\tau_l)}<v^*<1.$$

Next, substituting $0<u^*<1$ into $\dot{p}=0$ in Equation 11a yields $(1-\tau_k)r(u^*,p)=w(u^*,p)/p>0$, which determines a unique p^*. Moreover, substituting p^* into $\dot{s}=0$ in Equation 11b leads to

$$(1-\tau_k)r(u(p^*,z),p^*)-\rho-\varsigma=\sigma\left[\frac{1-u(p^*,z)}{\theta}\frac{w(u(p^*,z),p^*)}{p}-\varsigma\right]>0,$$

and determines z^*. Finally, $(1-\tau_k)r^*(u^*,p^*)=w^*(u^*,p^*)/p>0$ in Equation 11a and $\dot{z}=0$ in Equation 10c lead to

$$s^*=z^*\left(\frac{(1-g)v}{\alpha(1-\tau_k)}-\frac{1-u}{\theta}\right)(1-\tau_k)r^*,$$

which is positive under Condition S. Therefore, there exists a unique BGP.

Finally, using Equation 11a and Equation 11b yields

$$(1-\tau_k)r^*=\frac{\theta}{\theta-\sigma(1-u^*)}\rho>\rho,$$

as $1-\theta/\sigma<u^*<1$. Therefore, the economic growth rate in Equation A7 is positive.

Appendix C: Ruling Out the Case with Three Negative Eigenvalues

The determinant of the Bordered Hessian is

$$\begin{aligned}BJ=&\left\{\frac{(1-g)p^*z^*}{\alpha(1-\tau_k)}\left(\frac{\partial[(1-\tau_k)r]}{\partial p}-\frac{\partial(w/p)}{\partial p}\right)\frac{\partial u}{\partial z}\frac{v^*(1-v^*)}{u^*(1-u^*)}\right\}\\&+\left\{\frac{p^*z^*}{r^*(1-\tau_k)^2}\left(\frac{(1-g)v^*}{\alpha(1-\tau_k)}-\frac{1-u^*}{\theta}\right)\frac{\partial u}{\partial z}\left(\frac{\partial[(1-\tau_k)r]}{\partial p}\Sigma_4-\frac{\partial(w/p)}{\partial p}\right)\Sigma_3\right\}+\left\{\frac{p^*s^*}{\Gamma_1 z^*}(\Sigma_1-\Sigma_2)\right\}\\&+\left\{\frac{r^*(1-\tau_k)}{\theta}\frac{\partial u}{\partial z}\left[s^*+p^*z^*\frac{\partial[(1-\tau_k)r]}{\partial p}-\frac{\partial(w/p)}{\partial p}\right]\right\}+\left\{\frac{s^*}{(1-\tau_k)^2}\frac{\partial u}{\partial z}\left[\frac{\Sigma_3}{\sigma}-\frac{(1-u^*)\Sigma_4}{\theta}\right]\right\},\end{aligned} \tag{A11}$$

where

[10] Condition S is easily met if σ is larger than 1. For example, when $\tau_k=\tau_l=0.2$, $\gamma=\delta=.08$, $\varepsilon=\xi=.05$, $\alpha=.3$, $\beta=.57$, $\eta=.2$, and $\theta=.67$, both production functions satisfy constant returns to scale. Then, Condition S is $\sigma>0.8391$, which is satisfied as σ is generally larger than 1.

$$\frac{\partial r}{\partial p} = \frac{-(1-\tau_k)r^*(\beta+\varepsilon-\gamma\zeta)}{p^*\Lambda_1},$$

$$\frac{\partial(w/p)}{\partial p} = \frac{(1-\tau_k)r^*[\eta(1-r)+\delta(\alpha-\psi)]}{p^*\Lambda_1},$$

and $\partial u/\partial z$ is as defined in Equation 8.

In the following, we show $BJ < 0$ under the case with $DetJ > 0$. Recall that $DetJ < 0$ requires the condition $\Gamma_1 > 0$. Consider the following sets of conditions:

Condition D1: $\delta \geq \gamma$,
Condition D2: $\alpha > \eta$,
Condition D3: $(1-\tau_k)r^* \geq \frac{\Lambda_1}{\Lambda_2}$. *QED.*

Under Conditions D1–D3, *BJ* must be negative for the following reasons: While the first term in Equation A11 is negative under Conditions D1 and D2, the second term is negative under Condition S, D1 and D2, and Assumption 1(*iii*). Moreover, Assumption 1(*iii*) implies $[1-\psi(v^*+u^*)] \geq \psi(2-v^*-u^*)$, that, with Conditions D1 and D2, leads to $\Sigma_1 < 0$ and $\Sigma_2 > 0$, implying a negative value for the third term in Equation A11. Under Condition D3 and $C^* < K^*$, we obtain $C^* < K^* r^* \Lambda_2/\Lambda_1$, and, hence, the fourth term in Equation A11 is negative. Further, in a BGP, $\dot{s}/s = 0$ in Equation 12b is rewritten as

$$\frac{1}{\sigma}(1-\tau_k)r(p,z) - \frac{1}{\theta}\frac{[1-u(p,z)]w(p,z)}{p} = \frac{\rho+\zeta}{\sigma}. \tag{A12}$$

Differentiating Equation A12 with respect to u yields

$$\frac{1}{\sigma}\frac{\partial[(1-\tau_k)r]}{\partial u} - \frac{1-u}{\theta}\frac{\partial(w/p)}{\partial u} = -\frac{1}{\theta}\frac{w}{p} < 0. \tag{A13}$$

Note that the left-hand side of Equation A13 is equal to

$$(1-\tau_k)\left[\frac{\Sigma_3}{\sigma} - \frac{(1-u^*)\Sigma_4}{\theta}\right],$$

implying

$$\frac{\Sigma_3}{\sigma} - \frac{(1-u^*)\Sigma_4}{\theta} < 0.$$

Thus, the fifth term in Equation A11 is also negative. As a result, $BJ < 0$. However, $BJ < 0$ cannot be consistent with three negative eigenvalues. Hence, there is not the case with three negative eigenvalues.

Examining Conditions D1–D3, as $\delta < \gamma$ is a sufficient condition for $\Gamma_1 < 0$, Condition D1 is met when $\Gamma_1 > 0$. Condition D2 requires that the share of physical capital in the goods sector be greater than that in the education sector, a condition consistent with empirical evidence. Under Condition D1 and D2, $\Lambda_1 = \alpha(1-\delta) - \eta(1-\gamma) + \psi(\delta-\gamma) > 0$ and, hence, $\partial u/\partial z > 0$. Finally, Condition D3 requires a real net interest rate above a threshold level (i.e., $\Lambda_1/\Lambda_2 < 1$). The condition is easier to satisfy if the productivity coefficients A and B are large enough.

Indeed, the result that $BJ < 0$ is empirically plausible in a following parameter space whose range is wide: $\alpha \in (0.1, 0.4)$, $\eta \in (0.1, 0.4)$, $\gamma \in (0, 0.4)$, $\delta \in (0, 0.4)$, $\varepsilon \in (0, 0.2)$, $\xi \in (0, 0.2)$, $\tau_l \in (-1, 01)$, $\tau_k \in (-1, 1)$, and $g \in (0, 0.9)$.

Appendix D: Sector-Specific Externality with Both *K* and *H*

If we allow for both sector-specific externalities of *K* and *H*, the technologies become

$$Y = A(vk)^{\alpha}(uh)^{\beta}\left(\overline{vK}\right)^{\chi}\left(\overline{uH}\right)^{\varepsilon}\tilde{G}^{\gamma}, \quad K(0) > 0, H(0) > 0 \tag{A14}$$

given

$$X = B[(1-v)k]^{\eta}[(1-u)h]^{\theta}\left[\overline{(1-v)K}\right]^{\kappa}\left[\overline{(1-u)H}\right]^{\xi}\tilde{G}^{\delta}, \tag{A15}$$

where χ and κ are the degrees of sector-specific externalities associated with K in the goods sector and the education sector, respectively. All other notations and the environment are the same as those in the text. We have shown that Equation A4 is the same but is modified as

$$\begin{aligned}\theta\frac{X}{(1-u)H} &= \theta B(gA)^{\frac{\delta}{1-\gamma}}\left(\frac{\beta\eta(1-\tau_l)}{\alpha\theta(1-\tau_k)}\right)^{\eta+\kappa}\left(\frac{u}{1-u}\right)^{\delta}v^{\frac{\delta\gamma\psi}{1-\gamma}}u^{\frac{\delta\gamma\zeta}{1-\gamma}}(1-v)^{\delta\psi}(1-u)^{\delta\zeta}\left(\frac{v}{u}z\right)^{\frac{(\eta+\kappa)(1-\gamma)+\delta(\alpha+\chi-\psi)}{1-\gamma}}\\ &\equiv \frac{w(p,z)}{p} = \rho+\varsigma-\frac{\dot{\mu}}{\mu}.\end{aligned} \tag{A16}$$

With the help of Equations A4 and A16, we rearrange Equation A1b to obtain

$$\begin{aligned}&\frac{(1-\tau_l)\beta(Ag^{\gamma}g)^{\frac{1}{(1-\gamma)}}}{\theta B(gA)^{\delta/(1-\gamma)}[\beta\eta(1-\tau_l)/(\alpha\theta(1-\tau_k))]^{(\eta+\kappa)}}\left(\frac{u}{1-u}\right)^{-\delta}[v(u)]^{\frac{\psi\gamma(1-\delta)}{1-\gamma}}u^{\frac{\zeta\gamma(1-\delta)}{1-\gamma}}\\ &\times[(1-v(u))]^{-\delta\psi}[(1-u)]^{-\delta\zeta}\left(\frac{v(u)}{u}\right)^{\frac{\Lambda_1}{1-\gamma}} = z^{\frac{-\Lambda_1}{1-\gamma}}p,\end{aligned} \tag{A17}$$

where

$$\begin{aligned}\Lambda_1 &\equiv (\alpha+\chi)(\theta+\xi)-(\eta+\kappa)(\beta+\varepsilon)+\psi[\delta(1-\alpha-\chi)-\gamma(1-\eta-\kappa)]+\zeta[\gamma(\eta+\kappa)-(\alpha+\chi)\delta]\\ &\equiv (\alpha+\chi)(1-\delta)-(\eta+\kappa)(1-\gamma)-\psi(\gamma-\delta).\end{aligned} \tag{A18}$$

The relative capital intensity from the social perspective between the two sectors now becomes $\Delta^s = (\alpha+\chi)(\theta+\xi) - (\eta+\kappa)(\beta+\varepsilon)$.

Moreover, after the transformation, while $r(p, z)$ is the same, $w(p, z)/p$ now becomes

$$\frac{w(p,z)}{p} = \Pi p^{\frac{\eta(1-\gamma)+\delta(\alpha-\psi)}{\Lambda_1}}\left[\frac{u(p,z)[1-v(u(p,z))]^{\psi}}{[1-u(p,z)]^{(1-\zeta)}}\right]^{\frac{\delta(\alpha+\chi-\gamma\psi)}{\Lambda_1}}[v(u(p,z))]^{\frac{-\gamma\psi(\eta+\kappa-\delta\psi)}{\Lambda_1}}[u(p,z)]^{\frac{-\gamma\zeta(\eta+\kappa-\delta\psi)}{\Lambda_1}} \geq 0, \tag{A19}$$

where

$$\Pi = \theta B(Ag)^{\frac{\delta}{1-\gamma}}\left(\frac{\beta\eta(1-\tau_l)}{\alpha\theta(1-\tau_k)}\right)^{\eta+\kappa}\Phi^{\frac{-(\eta+\kappa)(1-\gamma)-\delta(\alpha+\chi-\psi)}{\Lambda_1}}.$$

We have shown that the dynamic equilibrium system is exactly the same as the expressions in Equations 11a–11c, except for the expression $w(p, z)/p$ in Equations 11a–11c, modified as Equation A19. Therefore, the structure of the dynamic equilibrium system with sector-specific externalities of H and K here is the same as that in the text, with sector-specific externalities of H emerging in the technology.

References

Aschauer, D. A. 1989. Is public expenditure productive? *Journal of Monetary Economics* 23:177–200.

Barro, R. J. 1990. Government spending in a simple model of endogenous growth. *Journal of Political Economy* 98:S103–S125.

Barro, R. J., and X. Sala-i-Martin. 1992. Public finance in models of economic growth. *Review of Economic Studies* 59:645–61.

Barro, R. J., and X. Sala-i-Martin. 1995. *Economic growth.* Singapore: McGraw-Hill.

Ben-Gad, M. 2003. Fiscal policy and indeterminacy in models of endogenous growth. *Journal of Economic Theory* 108:322–44.

Benhabib, J., and R. Farmer. 1996. Indeterminacy and sector specific externalities. *Journal of Monetary Economics* 37:397–419.

Benhabib, J., Q. Meng, and K. Nishimura. 2000. Indeterminacy under constant returns to scale in multisector economics. *Econometrica* 68:1541–8.

Benhabib, J., and K. Nishimura. 1998. Indeterminacy and sunspots with constant returns. *Journal of Economic Theory* 81:58–96.

Benhabib, J., and R. Perli. 1994. Uniqueness and indeterminacy: On the dynamics of endogenous growth. *Journal of Economic Theory* 63:113–42.

Bond, E. W., P. Wang, and C. K. Yip. 1996. A general two-sector model of endogenous growth with human and physical capital: Balanced growth and transitional dynamics. *Journal of Economic Theory* 68:147–73.

Cazzavillan, G. 1996. Public spending, endogenous growth, and endogenous fluctuations. *Journal of Economic Theory* 71:394–415.

Chen, B.-L. 2006. Public capital, endogenous growth and endogenous fluctuations. *Journal of Macroeconomics* 28:768–74.

Chen, B.-L. 2007. Factor taxation and labor supply in a dynamic one-sector growth model. *Journal of Economic Dynamics and Control* 31:3941–64.

Chen, B.-L., and S.-F. Lee. 2007. Congestible public goods and local indeterminacy: A two-sector endogenous growth model. *Journal of Economic Dynamics and Control* 31:2486–518.

Edwards, J. H. Y. 1990. Congestion function specification and the 'publicness' of local public good. *Journal of Urban Economics* 27:80–96.

Fisher, W. H., and S. J. Turnovsky. 1998. Public investment, congestion, and private capital accumulation. *Economic Journal* 108:399–413.

Futagami, K., Y. Morita, and A. Shibata. 1993. Dynamic analysis of an endogenous growth model with public capital. *Scandinavian Journal of Economics* 95:605–25.

Glomm, G., and B. Ravikumar. 1997. Productive government expenditures and long-run growth. *Journal of Economic Dynamics and Control* 21:183–204.

Gramlich, E. M. 1994. Infrastructure investment: A review essay. *Journal of Economic Literature* 32:1176–96.

Hu, Y., R. Ohdori, and K. Shimomura. 2005. *Indeterminacy in a two-sector endogenous growth model with productive government spending*. Discussion Paper Series No. 149, Research Institute for Economics and Business Administration, Kobe University.

Jones, L. E., R. E. Manuelli, and P. E. Rossi. 1993. Optimal taxation in models of endogenous growth. *Journal of Political Economy* 101:485–517.

Kehoe, T. 1991. Computation and multiplicity of equilibria. In *Handbook of mathematical economics.* Volume 4, edited by W. Hildenbrand and H. Sonnenschein. Amsterdam: North-Holland.

Lucas, R., Jr. 1988. On the mechanics of economic development. *Journal of Monetary Economics* 22:3–42.

Lynde, C., and J. Richmond. 1993. Public capital and long-run costs in UK manufacturing. *Economic Journal* 103:880–93.

McMillan, M. L. 1989. On measuring congestion of local public goods. *Journal of Urban Economics* 26:131–7.

Mino, K. 2001. Indeterminacy and endogenous growth with social constant returns. *Journal of Economic Theory* 97:203–22.

Mino, K. 2004. Human capital formation and patterns of growth with multiple equilibria. In *Human capital, trade and public policy in rapidly growing economies: From theory to empirics*, edited by M. Boldrin, B. Chen, and P. Wang. Aldershot, UK: Edward Elgars, pp. 42–64.

Mulligan, C., and X. Sala-i-Martin. 1993. Transitional dynamics in two-sector models of endogenous growth. *Quarterly Journal of Economics* 108:737–73.

Palivos, T., C. K. Yip, and J. Zhang. 2003. Transitional dynamics and indeterminacy of equilibria in an endogenous growth model with public input. *Review of Development Economics* 7:86–98.

Park, H., and A. Philippopoulos. 2004. Indeterminacy and fiscal policies in a growing economy. *Journal of Economic Dynamics and Control* 28:645–60.

Raurich, X. 2001. Indeterminacy and government spending in a two-sector model of endogenous growth. *Review of Economic Dynamics* 4:210–29.

Raurich, X. 2003. Government spending, local indeterminacy and tax structure. *Economica* 70:639–53.

Rebelo, S. 1991. Long-run policy analysis and long-run growth. *Journal of Political Economy* 99:500–21.

Starrett, D. A. 1988. *Foundations of public economics.* New York: Cambridge University Press.

Turnovsky, S. J. 1996. Optimal tax, debt, and expenditure policies in a growing economy. *Journal of Public Economics* 60:21–44.

Turnovsky, S. J. 2000. Fiscal policy, elastic labor supply, and endogenous growth. *Journal of Monetary Economics* 45:185–210.