Public capital, endogenous growth, and endogenous fluctuations

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Abstract

Cazzavillan [Cazzavillan, G., 1996. Public spending, endogenous growth, and endogenous fluctuations. Journal of Economic Theory 71, 394–415] studies a discrete-time, one-sector endogenous growth model with a flow of publicly enjoyable goods and productive services financed through income taxation. He demonstrates how equilibrium paths are indeterminate, for a large range of the consumption externality of public spending. This study extends [Cazzavillan, 1996] by considering an otherwise identical production function, except with public capital stock as an input. The results support the robustness of multiple growth paths even in a one-sector growth model with public capital stock, and modify the set of the consumption externality of public spending, in determining growth dynamics in a similar model with non-accumulated public spending.
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1. Introduction

Cazzavillan (1996) extends the standard, continuous-time, one-sector endogenous growth model in Barro (1990) into a discrete-time model. In this type of model, representative agents are infinitely lived, who hold the shares of the representative producers, and a

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government provides the private sector a flow of publicly enjoyable goods and services financed through income taxation. In particular, the public spending *flows* increase not only representative agents' utility, but also representative producer's productivity. Under his framework, Cazzavillan shows that the equilibrium growth paths are indeterminate, for a large range of the consumption externality of public expenditure. This result is significant as it establishes endogenous growth fluctuations without relying on increasing returns or externality in production (e.g., Benhabib and Farmer, 1994; Benhabib and Perli, 1994; Farmer and Guo, 1994), or search frictions (e.g., Chen et al., 2002).

In solving his model's equilibrium, Cazzavillan postulates public spending *flow* as initially given, although Barro (1990) treats it as endogenous. In a similar continuous-time, one-sector growth model with public spending *flows* affecting utility and production where the initial public spending flow is treated as endogenous, Park and Philippopoulos (2002) show that the Barro model exhibits no transitional dynamics, and thereby no endogenous fluctuations in growth. In order to generate transitional growth dynamics, the Barro growth model has been modified by Futagami et al. (1993) and Turnovsky (1997) to allow for public capital *stock* in production. Since both private investment and public spending can be accumulated into stocks, the otherwise one-sector Barro model behaves like a two-sector model, and therefore the model exhibits transitional dynamics. Although models by Futagami et al. and Turnovsky allow for the accumulation of public capital and consider public capital stock in production, none of them incorporate the consumption externality of public spending flows. As a result, these models display no indeterminate growth paths.

The purpose of this paper is to extend Cazzavillan into a model that, while maintaining public spending flows in utility, allows for public spending to accumulate public capital and to consider public capital stock in production. With this modification, I thus show the plausibility of indeterminate equilibrium paths in this model, and quantitatively assess the set of the consumption externality of public spending in determining the associated equilibrium dynamics. More specifically, I find that the set of consumption externality of public spending for generating multiple equilibrium paths shrinks tremendously, and there exists a set of the externality that leads to unstable equilibrium paths. Moreover, while the unstable paths and all the unique equilibrium paths are monotonic, the multiple equilibrium paths could be cyclical and monotonic. Finally, there is a possibility of a Hopf bifurcation, and thus the equilibrium dynamics could exhibit limit cycles.

Intuitively, when the consumption externality of public spending falls into a range so that the interaction between private capital and public capital flows is strong enough, the model generates multiple growth paths as shown in Cazzavillan (1996). By requiring a public spending flow to accumulate public capital stock before it affects production, as opposed to a public spending flow directly affecting production, more public capital interacts with private capital in production. This then increases the (pareto) complementarity between private and public capital. With a stronger complementarity in production, in order to yield multiple growth paths, the required contribution of public consumption externality is smaller. Therefore, the set of consumption externality of public spending for deterministic growth fluctuations shrinks.

As developed below, the main body of this paper is in Section 2, which sets up the model, conducts its optimization, steady-state equilibrium and dynamic equilibrium analysis, and carries out simulations to find the set of the consumption externality of public spending for an indeterminate, a unique and an unstable equilibrium, and a Hopf bifurcation. Finally, Section 3 presents conclusions.

2. The model

Cazzavillan's (1996) model is a discrete-time variant of Barro (1990), in which there is an infinite-lived, representative household/producer. The agent decides its consumption and savings/capital accumulation, in order to maximize its discounted present value of lifetime utility. There is also government, which uses income taxes to finance public spending. Public spending affects both households' utility and producers' productivity. The Cazzavillan's parametric version is employed here to illustrate the ideas.

The government collects income taxes and uses the revenue to finance its expenditure flows. A public spending flow affects households' utility instantaneously, but, different from Cazzavillan (1996), the public spending flow affects production only if it is accumulated into an infrastructure stock. In order to closely follow his framework, the model in this paper is simplified by assuming a one-hundred percent depreciation rate for public capital stock, so that it is like a public spending flow entering the production in his model, albeit occurring in the next period. When we denote τ as a flat income tax rate, the government budget constraint and the public capital accumulation are thus:

$$G_{t+1} = g_t = \tau Y_t, \quad G_0 > 0 \text{ given}, \tag{1}$$

where g is a public spending *flow*, G is public infrastructure *stock*, and Y is output.

The felicity function and the output production function are, respectively:

$$u(g_t, C_t) = \frac{g_t^\eta C_t^v}{g_t^v + C_t^v} = \frac{g_t^{\eta-v} C_t^v}{1 + (C_t/g_t)^v}, \tag{2}$$

$$Y_t = A K_t^\alpha G_t^{1-\alpha}, \quad K_0 > 0 \text{ given}, \tag{3}$$

where C_t is a consumption flow, and K_t is a private capital stock.

I should mention that the first part of (1) and (3) is the only departure of this model from Cazzavillan: public spending flows accumulate the infrastructure in (1); and the public infrastructure, rather than the public spending flows, enters the output production function in (3).

While the felicity function is homogeneous of degree η, the production function is homogeneous of degree one. I assume that productivity parameter is $A > 0$, share of private capital stock in the production is $\alpha \in (0,1)$, the degree of the public spending externality on consumption is $\eta - v > 0$, and the contribution of unit consumption to utility is $v \in (0,1)$. Although in general it suffices to allow for $\eta > 0$, it is necessary to impose $\eta > 1$ under this parametric model.

Denoting $\delta > 0$ as the depreciation rate of private capital stock, a representative household's budget constraint is:

$$K_{t+1} = (1-\tau)Y_t - C_t + (1-\delta)K_t. \tag{4}$$

Assuming a discounted factor $\beta \in (0,1)$, the representative agent, who takes g_t, G_t and τ as given, chooses consumption and savings to maximize his discounted present value of lifetime utility, given his budget constraint (4). Denoting $k_t \equiv K_t/G_t$, and $c_t \equiv C_t/G_t$, the optimization leads to:

$$\beta^t v \frac{g_t^{\eta+v} C_t^{v-1}}{(g_t^v + C_t^v)^2} = q_t, \tag{5a}$$

$$q_{t+1}[(1-\tau)A\alpha k_{t+1}^{\alpha-1} + (1-\delta)] = q_t, \tag{5b}$$

where (5a) equates marginal utility of current consumption to marginal cost of current consumption, and (5b) is the Euler equation governing optimal private capital accumulation.[1]

Market equilibrium is governed by production technology (3), optimization conditions (5a) and (5b), household budget constraints (4), and government budget constraints (1). To solve the equilibrium, it is easier if I transform the equilibrium condition into a planar system (k_t, c_t). This can be done first, by combining the second equality in (1), (5a), (5b) with production function (3), and second, by dividing both sides of (4) by G_{t+1}, with the use of the first equality in (1) and production function (3), to yield respectively:

$$(\tau A)^{\eta-1} \frac{\beta k_{t+1}^{\alpha(\eta+v)} c_{t+1}^{v-1}}{[(\tau A)^v k_{t+1}^{\alpha v} + c_{t+1}^v]^2} = \frac{1}{(1-\tau)A\alpha k_{t+1}^{\alpha-1} + (1-\delta)} \frac{k_t^{\alpha(1+v)} c_t^{v-1}}{[(\tau A)^v k_t^{\alpha v} + c_t^v]^2}, \tag{6a}$$

$$k_{t+1} = \frac{1-\tau}{\tau} - \frac{1}{\tau A} c_t k_t^{-\alpha} + \frac{1-\delta}{\tau A} k_t^{1-\alpha}. \tag{6b}$$

2.1. Balanced-growth paths

In a balanced-growth path, C_t, K_t, and G_t all grow at the same rate. As a result, $k_{t+1} = k_t = k$ and $c_{t+1} = c_t = c$ are constant for all t. Then, (6a) and (6b) become:

$$(\tau A)^{\eta-1}\beta(1-\delta)k^{\alpha(\eta-1)} = 1 - (1-\tau)\tau^{\eta-1}A^\eta\alpha\beta k^{\alpha\eta-1}, \tag{7a}$$

$$c = (1-\tau)Ak^\alpha + (1-\delta)k - \tau Ak^{1+\alpha} \tag{7b}$$

System (7a) and (7b) is recursive; while (7a) determines steady state k, (7b) determines steady state c. Below we show there exists a unique balanced-growth path $\{c, k\}$. The left-hand side of (7a) increases in k from the origin, given $\eta > 1$, whereas the right-hand side decreases in k from intercept 1 if $\eta > 1/\alpha$. Thus, if $\eta > 1/\alpha$, there exists a unique steady state k. Moreover, when $1 < \eta < 1/\alpha$, the right-hand side of (7a) increases in k. Then there exists $\eta^* \in (1, 1/\alpha]$ such that, in a plane with k as the horizontal axis, the locus of the right-hand side of (7a) passes the locus of the left-hand side from above. Therefore, if $\eta^* \leqslant \eta \leqslant 1/\alpha$, there also exists a unique steady state k. Finally, substituting steady state k into (7b) yields the unique steady state c. The economic growth rate, using (4), is then $\gamma_{t+1} \equiv Y_{t+1}/Y_t - 1 = \gamma = K_{t+1}/K_t - 1 = G_{t+2}/G_{t+1} - 1 = (1-\tau)A - \delta - c$. Therefore, there exists a unique balanced-growth path $\{k,c\}$ when $\eta \geqslant \eta^* \in (1, 1/\alpha]$.

2.2. Transitional growth dynamics

As there exists a unique balanced-growth path $\{k,c\}$ when $\eta \geqslant \eta^*$, the economy's transitional dynamics can be obtained by taking a linear, Taylor's expansion of (6a) and (6b) around the unique balanced-growth path as follows:

$$\begin{pmatrix} k_{t+1} - k \\ c_{t+1} - c \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} k_t - k \\ c_t - c \end{pmatrix} \tag{8}$$

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[1] Transversality condition $\lim_{t\to\infty} q_t K_t = \lim_{t\to\infty} v\beta^t k_t^{\alpha(\eta-v)+1} c_t^{v+\eta-1}/[(\tau A)^v k_t^{\alpha v} + c_t^v]^2 = 0$ is also required.

in which $a_{11} \equiv \frac{1}{\tau A k^\alpha}\left[\alpha \frac{c}{k} + (1-\alpha)(1-\delta)\right] > 0, a_{12} \equiv -\frac{1}{\tau A k^\alpha} < 0, a_{21} \equiv \frac{Z-S}{X}a_{11} - Z,$ and $a_{22} \equiv \frac{Z-S}{X}a_{12} + 1,$ where $Z \equiv \frac{\alpha}{k}\left[\eta + v - 2v\frac{(\tau A)^v k^{\alpha v}}{(\tau A)^v k^{\alpha v}+c^v}\right] > 0, S = \frac{(1-\tau)A\alpha(1-\alpha)k^{\alpha-2}}{(1-\tau)A\alpha k^{\alpha-1}+(1-\delta)} > 0$ and $X = \frac{1-v}{c} + \frac{2vc^{v-1}}{(\tau A)^v k^{\alpha v}+c^v} > 0.$

Given that system (8) has a jump variable (c) and a state variable (k), the economy possesses a unique equilibrium saddle path leading the economy to move toward the unique steady state from any initial state, if one of the real parts of the two roots of the Jacobean matrix in (8) lies inside the unit circle, and the other lies outside the unit circle. If both real parts of the two roots of the Jacobean matrix lie outside the unit circle, then there exists no equilibrium path leading the economy to move toward the unique steady state, and the economy is unstable. Finally, if the real parts of the two roots of the Jacobean matrix lie inside the unit circle, the economy has a continuum of equilibrium paths leading the economy to move toward the unique steady state, and therefore the equilibrium path is indeterminate. Moreover, if the roots involve an imaginary part, then the equilibrium paths are cyclical; otherwise, they are monotonic.

When the real parts of the two eigenvalues lie inside the unit circle, dynamic paths toward the unique balanced-growth path are indeterminate for c_t and k_t. As a result, the growth rates of capital and public capital, $K_{t+1}/K_t - 1 = (1-\tau)Ak_t^{-(1-\alpha)} - c_t - (1-\delta)$ and $G_{t+1}/G_t - 1 = Y_t/Y_{t-1} - 1 = \gamma_t$, are indeterminate, and so is the economic growth rate, $\gamma_{t+1} = Y_{t+1}/Y_t - 1 = (k_{t+1}/k_t)^\alpha G_{t+1}/G_t - 1 = (k_{t+1}/k_t)^\alpha(\gamma_t + 1) - 1.$

2.3. Numerical analysis

In order to quantitatively assess the transitional dynamics properties, Cazzavillan (1996) sets $A = 0.4575$, $\alpha = 0.667$, $\beta = 0.96$, $\delta = 0.04$, $v = 0.8$ and $\tau = 0.25$, and obtains a continuum of equilibrium paths for $\eta \in (1.107, 33.1)$, and a unique equilibrium path for $\eta > 33.1$. No unstable equilibrium is obtained for $\eta > 1.107$. I employ the same set of parameter values to quantify the dynamic properties for different values of η. The results are reported in Table 1.

From Table 1, several interesting findings are summarized here. First, the set of η that leads to indeterminate equilibrium growth paths shrinks tremendously to $\eta \in [1.105, 5.182]$, whereas the set that yields a unique equilibrium path expands into $\eta > 5.857$. Second, there exists a range of $\eta \in (5.182, 5.857)$ under which the equilibrium path is unstable. The set of η leading to indeterminate growth paths is much smaller than that under Cazzavillan's setup, reducing the largest degree of public consumption externality, $\eta - v$, from 32.3 in Cazzavillan to 4.38. Although 4.38 is still large and may be empirically implausible, it is 13.5% that Cazzavillan. Given that our model is only a small

Table 1
Dynamic properties

η	Equilibrium path	Equilibrium path is cyclical or monotonic
(1.105, 5.182)	Indeterminate	Cyclical/monotonic
(5.182, 5.857)	Unstable	Monotonic
(5.857, ∞)	Unique	Monotonic

Note: This table is simulated based on Eqs. (7a), (7b) and (8), with $A = 0.4575$, $\alpha = 0.667$, $\beta = 0.96$, $\delta = 0.04$, $v = 0.8$ and $\tau = 0.25$. *Mathematica* is used to conduct simulation.

departure from Cazzavillan's setup, the reduction in the set of public consumption externality by 86% is a tremendous shrink. In order to reduce the degree of externality to an empirically plausible range, a further modification of the Barro-type model is necessary. In the Concluding Section, we will discuss possible directions of extensions.

2.4. Hopf Bifurcations

Different from the one-dimensional system going through a Flip bifurcation in Cazzvillan, our planar system does not goes through a Flip bifurcation. We cannot find a single real eigenvalue on the boundary of the unit circle with value −1. Consequently, the dynamics of our economic system will not become period-doubling. As our dynamic system is two-dimensional and there is a set of η with eigenvalues that are complex conjugates, it is possible for the dynamic to exhibit a Hopf bifurcation. If this situation appears, the dynamics of the economic system display limit cycles. We investigate such a possibility.

Denote the two eigenvalues as $\lambda_1 = \omega + \theta i$ and $\lambda_2 = \omega - \theta i$. In order to obtain a Hopf bifurcation, the following three conditions need to satisfy:

1. $\Delta = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12}) < 0$,
2. $|\omega| = \left| \frac{a_{11} + a_{22}}{2} \right| < 1$,
3. $\lambda_1 \lambda_2 = \omega^2 + \theta^2 = a_{11}a_{22} - a_{21}a_{12} = 1$.

Using the set of parameter values used in Section 2.3, all the above three conditions are met when $\eta = 1.3061$.[2] Therefore, there is a possibility for the equilibrium dynamics to exhibit limit cycles.

3. Concluding remarks

This paper has extended Cazzavillan's (1996) one-sector, endogenous growth model into one with public capital stock entering the production. I analyzed the equilibrium dynamics and quantified the parameter space of the consumption externality of public expenditure for indeterminate, unique and unstable equilibrium growth paths. The results support the robustness of multiple equilibrium growth paths even in a one-sector growth model with public capital stock, and also modify the set of the consumption externality of public spending in determining growth dynamics in a similar model with non-accumulated public spending.

Although the set of public consumption externality for indeterminate equilibria shrinks under my setup, the required degree of public consumption externality is still large and is not within an empirically plausible range. For this very reason, Cazzavillan (1996) considers a stochastic model in his Section 5 so that stochastic growth fluctuations (sunspots) can arise under a smaller degree of public consumption externality. If my model is endowed with endogenous uncertainty, with a stronger interaction between public and private capital stocks in production, the set of public consumption externality for stochastic indeterminate equilibria is expected to be smaller than that in Cazzavillan.

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[2] The two eigenvalues are $0.761237 \pm 0.648473i$.

Alternatively, I may extend this model to a deterministic, multi-sector model in order to shrink the set of public consumption externality to an empirically plausible range. This direction of extension is in line with the studies on real business cycles and endogenous growth that emphasize the existence of indeterminate equilibria in the framework of deterministic models with increasing returns or external effects [see survey in Benhabib and Farmer, 1999]. Much effort has been devoted to reduce the degree of increasing returns or external effects in a one-sector model (e.g., Wen, 1998), a two-sector model (e.g., Gou and Harrison, 2002; Harrison, 2001), and a multisector model (e.g., Benhabib et al., 2000).

Acknowledgements

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