Non-separable Utilities and Aggregate Instability*

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Abstract

This paper studies an infinite-horizon two-sector growth model with sector-specific externalities and preferences that are non-separable between consumption and leisure. We find two main results. First, a larger income effect on the labor supply increases the possibility of macroeconomic instability. Second, a larger elasticity of the labor supply may increase or decrease the possibility of aggregate instability, depending on the intensity of the income effect.

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1 Introduction

A strand of the growth literature has shown that equilibrium indeterminacy can explain macroeconomic instability in the infinite-horizon growth model when sector specific externalities are considered. This literature has analyzed how the existence of aggregate instability depends on both the intensity of the income effect on the labor supply and on the value of the wage elasticity of the labor supply. It has obtained different results in the one-sector and two-sector growth models.

In a one-sector growth model, Jaimovich (2008) studied aggregate instability or indeterminacy with a non-separable utility function. The utility features different intensities of the income effect on the labor supply and nests, as special cases, the two classes of utility functions most widely used in the business cycle literature, those characterized in King, Plosser, and Rebelo (1988) and in Greenwood, Hercowitz, and Huffman (1988).[1] He found that, regardless of the degree of increasing returns to scale, the one-sector growth model does not generate aggregate instability when there is no income effect on the labor supply. Similarly, in a one-sector growth model, Meng and Yip (2008) found that the presence of labor-supply income effects is required for indeterminacy to occur.

1 Jaimovich and Rebelo (2009) also consider this class of utility functions to show that labor-supply income effects lie at the heart of the ability to generate aggregate and sectoral comovement, which is a central feature of business cycles. In particular, they can generate aggregate comovement in the presence of moderate labor-supply income effects and low labor-supply income effects are essential to generate sectoral comovement.

These results are different from the existence of indeterminacy in one-sector growth models with sufficiently strong increasing returns proved by Benhabib and Farmer (1994) and Farmer and Guo (1994). These latter authors use a utility function that is characterized by the presence of income effects on the labor supply. Moreover, they showed that a higher elasticity of the labor supply increases the possibility of aggregate instability.[2]

In a two-sector growth model with no income effect on the labor supply, Guo and Harrison (2010) obtained a very different result, as they proved the existence of equilibrium indeterminacy. Specifically, they studied a class of two-sector models à la Benhabib and Farmer (1996) and Harrison (2001), where the technologies in the two sectors are differentiated by the introduction of sector-specific externalities and the two sectors are identified as the investment and consumption sectors. These authors adopted the non-separable utility function used in Greenwood, Hercowitz and Huffman (1988). The labor supply obtained from this utility function does not exhibit income effects. They showed that equilibrium indeterminacy emerges when sufficiently strong investment externalities are present. They also numerically showed for a logarithmic utility function that indeterminacy is more likely for low values of the labor supply elasticity (henceforth, LSE). Dufourt, Nishimura and Venditti (2015) proved this result analytically under a more general specification of no-income effect

[2] See also Lloyd-Braga et al. (2006) and Coury and Wen (2009).

preferences. These results are very different from the finding that indeterminacy is more likely for high values of LSE, which is obtained in models with income effects on the labor supply (e.g., Benhabib and Farmer, 1996).

The aforementioned literature has obtained very different results concerning the effect on aggregate instability of the intensity of the income effect on the labor supply and of the LSE. The purpose of this paper is to contribute to this literature by clarifying these effects. To this end, we consider a two-sector growth model with sector specific externalities à la Benhabib and Farmer (1996) and a non-separable utility function that introduces a varying degree of income effects on the labor supply.[3] At a limiting case, when the income effect on the labor supply is zero, this utility function encompasses the preferences considered by Guo and Harrison (2010). In another limiting case, when the income effect is the largest, the income elasticity of the labor supply equals the LSE. In non-limiting cases, the income elasticity is smaller than the LSE.

We obtain two main findings. First, the presence of the income effect on the labor supply increases the possibility of indeterminacy, as it reduces the minimum value of the intensity of the investment externality needed for indeterminacy. Second, the relationship between the LSE and indeterminacy depends on the intensity of the income effect. If the income effect is small, a decrease in the LSE

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3 The utility function is simpler than the one used in Jaimovich (2008) and Jaimovich and Rebelo (2009) which, in the case with income effects on the labor supply, involves a consumption habit that accumulates from past consumption. We do not consider consumption habits in our analysis.

reduces the minimum value of the intensity of the investment externality needed for indeterminacy and, hence, increases the possibility of indeterminacy. In contrast, if the income effect is large, a decrease in the LSE decreases the possibility of indeterminacy. Thus, the relationship in Guo and Harrison (2010) arises only when the income effect is small. This second result is particularly interesting as evidence on the value of the intensity of the income effect is not conclusive.[4]

The intuition on these two results is based on expectations on a higher interest rate that cause a reduction in today's consumption and an increase in tomorrow's consumption. We show that these changes increase the marginal rate of substitution (henceforth, MRS) between today's and tomorrow's consumption. As in the consumption Euler equation the MRS equals the marginal product of capital (henceforth, MPK), the MPK must increase to explain equilibrium indeterminacy. This requires a sufficiently large intensity of the investment externality. The presence of income effects on the labor supply limits the increase in the MRS, which reduces the minimum value of the intensity of the investment externality needed for indeterminacy. This explains the first result.

To understand the second result, it must be taken into account that the MRS also depends on the labor supply and, thus, the Euler equation depends on the

4 Business cycle literature has considered that preferences with no income effects provide a better fit to explain business cycles facts (Greenwood, Hercowitz, and Huffman , 1988). However, other authors provide estimates of a large positive income effects (Khan and Tsoukalas, 2011).

changes in the labor supply, which are obviously determined by the LSE and the income effect. When the income effect on the labor supply is small, a decrease in the LSE and an increase in the intensity of the investment externality both lead to smaller changes in the labor supply tomorrow. Thus, the lower the LSE, the lower the degree of increasing returns needed to keep the increase in the labor supply tomorrow small enough to satisfy the Euler equation. Conversely, when the income effect is large, increases in the LSE and increases in the intensity of the investment externality both lead to smaller changes in the labor supply tomorrow. Thus, the lower the LSE, the higher the degree of increasing returns needed to keep the increase in the labor supply tomorrow small enough to satisfy the Euler equation.

The analysis in this paper is related to different papers in the literature. On the one hand, it is related to those papers that study the relationship between the LSE and the possibility of indeterminacy. First, our results are related to the paper by Guo and Harrison (2010) and, in particular, our results clarify that the finding in Guo and Harrison (2010) on the effect that the LSE has on the possibility of indeterminacy arises only when the income effect is small. Second, our results are also related to Benhabib and Farmer (1996, Section 6) and Harrison (2001, Section 3), which also analyzed the effect of sector specific externalities in models with separable preferences and strong income effects on the labor supply. Their papers found that higher LSE increases the possibility of indeterminacy. We clarify that this result only occurs for large enough intensities of the income effect. Finally, we also remark that Harrison and Weder (2002) studied indeterminacy by extending the Benhabib and Farmer (1996) model to one that covers aggregate, sector-specific and factor-specific externalities. They analyzed the cases of one- and two-sector models. Our model is different from theirs as they adopt a utility function with indivisible labor which implies an infinite LSE.

On the other hand, there is a recent paper by Dufourt, Nishimura, Nourry and Venditti (2017) that, like our paper, also analyses local indeterminacy in a model with a varying degree of the income effect on the labor supply.[5] They show that local indeterminacy occurs for plausible values of the parameters in the two sector model with sector specific externalities. We attain the same conclusion in our numerical analysis. Moreover, they study how local indeterminacy depends on the value of the intertemporal elasticity of substitution (henceforth, IES) and other preference parameters.[6] In contrast, our paper studies how the minimum degree of the investment externality needed for indeterminacy depends on preference parameters. We find that a higher income effect increases the possibility of

5 Dufourt et al. (2017) considered a formulation of preferences based on the preferences introduced by Jaimovich (2008) and Jaimovich and Rebelo (2009). Contrary to the aforementioned papers, their specification of preferences does not introduce any consumption habit. This formulation of preferences has been initially introduced by Nourry, Seegmuller and Venditti (2013) and Abad, Seegmuller and Venditti (2017). In contrast to these authors, we consider different preferences for which the expression of the labor supply is simpler although it still exhibits a variable degree of income effects.

6 Dufourt et al. (2017) obtain two main results. First, with a larger income effect, a lower IES is needed for indeterminacy. Second, when the LSE is small, there exists a range of the IES such that indeterminacy arises.

indeterminacy through lowering the minimum degree of the investment externality required for indeterminacy. More importantly, we show that that the relationship between the LSE and the possibility of indeterminacy is not monotonous and it depends on the intensity of the income effect. This was not explored in Dufourt et al. (2017).

We organize this paper as follows. In Section 2, we set up the model and we obtain the equilibrium and the steady state. In Section 3, we study the transitional dynamics. In Section 4, we discuss the possibility of equilibrium indeterminacy. Finally, concluding remarks are offered in Section 5.

2 Model

2.1 Firms

The production side of the economy consists of two sectors. One of the sectors produces a consumption good, *c*, and the other sector produces an investment good, *y*. In each sector, output is produced by competitive firms using the following technologies:

$$y=\hat{A}_1\left(sk\right)^{\alpha}\left(uL\right)^{1-\alpha}, \tag{1a}$$

$$c=A_2\left(\left(1-s\right)k\right)^{\alpha}\left(\left(1-u\right)L\right)^{1-\alpha}, \tag{1b}$$

where $s\in(0,1)$ measures the fraction of capital, *k*, $u\in(0,1)$ measures the fraction of labor, *L*, employed in the investment sector, and $\hat{A}_1$ and A_2 measure, respectively, the total factor productivities in the investment and consumption sectors. The output elasticity of capital in both sectors is measured by $\alpha \in (0,1)$. Investment goods accumulate capital as follows:

$$\dot{k} = y - \delta k, \tag{2}$$

where $\delta > 0$ is the depreciation rate.

We assume that the total factor productivity in the investment sector is endogenous and depends on sector-specific externalities according to the following equation:

$$\hat{A}_1 = A_1 \left[\left(\overline{sk} \right)^{\alpha} \left(\overline{uL} \right)^{1-\alpha} \right]^{\theta},$$

where an upper bar means the average value of the variable in the economy and, $\theta \geq 0$ measures the strength of sector-specific externalities in the investment sector.

2.2 Households

Consider a unit measure of infinitely lived representative households characterized by the following utility function:

$$u(c,L) = \log\left(\frac{c^{\phi}}{\phi} - \beta \frac{L^{1+\chi}}{1+\chi} \right),\ \beta > 0,\ \phi \in [0,1] \text{ and } \chi \geq 0, \tag{3}$$

where c is consumption and L is the labor supply. We denote $\Delta \equiv \frac{c^{\phi}}{\phi} - \beta \frac{L^{1+\chi}}{1+\chi} > 0$. The assumptions $\phi \leq 1$ and $\chi \geq 0$ ensure the joint concavity with respect to consumption and leisure. The assumption $\phi \geq 0$ is needed to guarantee a welldefined utility function.

Let *ρ>0* be the subjective discount rate. The representative household maximizes the discounted sum of utilities, $\int_0^\infty e^{-\rho t} u(c, L)$, subject to (1a), (1b) and (2). Let *λ* be the co-state variable associated with (2) and, thus, it is the shadow price of investment goods. The first-order conditions with respect to *L*, *s*, *u* and *k* are, respectively (see Appendix A):

$$\frac{(1-\alpha)c^{\phi}-\beta L^{1+\chi}}{\Delta L}+\lambda(1-\alpha)\frac{y}{L}=0, \tag{4a}$$

$$\frac{\alpha c^{\phi}}{\Delta(1-s)}=\lambda\alpha\frac{y}{s}, \tag{4b}$$

$$\frac{(1-\alpha)c^{\phi}}{\Delta(1-u)}=\lambda(1-\alpha)\frac{y}{u}, \tag{4c}$$

$$\frac{\alpha c^{\phi}}{\Delta k}+\lambda(\alpha\frac{y}{k}-\delta)=\rho\lambda-\dot{\lambda}; \tag{4d}$$

with the transversality condition $\lim_{t\to\infty}\lambda k e^{-\rho t}=0.$

From using (1b) and assuming perfect competition, we obtain that the wage, *w,* in units of the consumption good is $w=(1-\alpha)c/[(1-u)L]$. Using (4a) and (4c), we obtain the labor supply:

$$L=\left(\frac{w}{\beta c^{1-\phi}}\right)^{\frac{1}{\chi}}. \tag{5}$$

The elasticity of the labor supply is $1/\chi$ and $(1-\phi)/\chi$ is the income elasticity of the labor supply. Thus, the parameter ϕ measures the strength of the income effect, with a smaller ϕ implying a larger income effect. In the limiting case of ϕ=1, there is no income effect. This case was studied by Guo and Harrison (2010). In the other limiting case, ϕ=0, the income effect is the largest and the elasticity of the income effect equals the elasticity of the labor supply. Benhabib and Farmer (1996) and Harrison (2001) studied this case in a model with a separable utility. We contribute to this literature by studying the case $\phi \in [0,1]$, which allows us to analyze how the intensity of the income effect affects the possibility of equilibrium indeterminacy.

2.3 Equilibrium

We proceed to obtain the equations characterizing the equilibrium. First, we divide (4b) by (4c) to obtain *s*=*u*. Next, we assume a symmetric equilibrium, implying that along an equilibrium path the average value of the variables coincides with the value of the variables at each firm. It follows that, along an equilibrium path, (1a) and (1b) simplify as follows:

$$y = A_1 s^{(1+\theta)} k^{\alpha(1+\theta)} L^{(1-\alpha)(1+\theta)}, \tag{6a}$$

$$c = A_2 (1-s) k^{\alpha} L^{1-\alpha}. \tag{6b}$$

To ensure that in equilibrium the production of investment goods exhibits diminishing returns to capital and labor, we assume $\theta<\alpha(1+\theta)<1$.

We next substitute the wage equation, (5) and *u*=*s* into the definition of Δ to obtain:

$$\Delta \equiv \frac{c^{\phi}}{\phi} - \beta \frac{L^{1+\chi}}{1+\chi} = \frac{c^{\phi}}{\phi} \frac{M}{(1-s)}, \tag{7}$$

where

$$M \equiv 1 - s - \frac{(1-\alpha)\phi}{(1+\chi)} > 0.$$

Moreover, if we combine (7) and (4b), we obtain:

$$y = \frac{\phi s}{\lambda M}. \tag{8}$$

From using the wage equation, (6b), *u*=*s* and (5), we rewrite the labor supply as:

$$L = \left[\frac{(1-\alpha)A_2^{\phi}}{\beta} (1-s)^{\phi-1} k^{\alpha\phi} \right]^{\frac{1}{\psi}} \equiv \hat{L}(s,k), \tag{9}$$

where $\psi \equiv 1 + \chi - \phi(1-\alpha) > 0$.

Note that the sectoral composition *s* affects the labor supply when there are income effects (ϕ<1). In this case, an increase in *s* reduces consumption, as the fraction of production factors employed in the production of consumption goods decreases. This reduction increases the labor supply because of the income effect.

Substituting (6a) and (9) into (8), we obtain:

$$(1-s)^{\frac{(1-\alpha)(1+\theta)(1-\phi)}{\psi}} = \frac{A_1}{\phi} \left(\frac{(1-\alpha)A_2^{\phi}}{\beta} \right)^{\frac{(1-\alpha)(1+\theta)}{\psi}} k^{\varepsilon} \lambda s^{\theta} M, \tag{10}$$

where $\varepsilon = (1+\theta)\alpha(1+\chi)/\psi > 0$. This equation implicitly defines the following function: $s=\hat{s}(\lambda,k)$. From using (10), the following lemma can be proved:

Lemma 1 $\frac{\partial\hat{s}}{\partial\lambda}>(<)0$ and $\frac{\partial\hat{s}}{\partial k}>(<)0$ if and only if $\Omega<(>)0,$ where

$$\Omega\equiv\psi(1-s)\left[(1+\theta)(1+B)M-s-M\right],$$

and

$$B\equiv\frac{(1-\phi)(1-\alpha)s}{\psi(1-s)}.$$

We proceed to obtain the system of differential equations governing the transition. First, from combining (8) with (4b) and (4d), we obtain:

$$\frac{\dot{\lambda}}{\lambda}=\rho+\delta-\frac{\alpha\phi}{\lambda kM}. \tag{11}$$

Next, from substituting (8) into (1), we obtain:

$$\frac{\dot{k}}{k}=\frac{s\phi}{\lambda kM}-\delta. \tag{12}$$

An equilibrium is a path of {*L*, *s*,λ, and *k*} that, given an initial stock of capital, solves the system of differential equations, (11) and (12), and satisfies (9), (10) and the transversality condition. A steady state is an equilibrium path along which variables remain constant.

Proposition 1 *There exists a unique steady state equilibrium. The steady state value of sectoral composition, s*, is*

$$s^* = \frac{\alpha\delta}{\rho+\delta} < 1.$$

The proof of this result follows from using the system of equations (11) and (12) evaluated at the steady state.

3 Indeterminacy with non-separable utilities

This section provides simple conditions for the existence of local indeterminacy with non-separable utilities. In Appendix B, we obtain the determinant and the trace of the Jacobian matrix associated with the system of differential equations characterizing the transitional dynamics. They are, respectively:

$$Det = \frac{\delta\psi\xi(1-s^*)}{1+\chi}\frac{(1-\varepsilon)}{\Omega},$$

and

$$Tr = -M\frac{\psi(1-s^*)}{\Omega}\left\{\delta\varepsilon-(\rho+\delta)\left[B+\theta(1+B)\right]\left(1-\frac{\delta\varepsilon}{\rho+\delta}\right)\right\}-\delta(1-\varepsilon),$$

where

$$\xi \equiv (1-\alpha)\delta(1+\chi-\phi)+\rho\left[1+\chi-(1-\alpha)\phi\right] > 0.$$

Since the two-dimensional dynamical system includes a predetermined variable and a jump variable, the equilibrium exhibits indeterminacy when the Jacobian matrix has two roots with negative real parts. This occurs if *Det*>0 and

Tr<0.

To state our main proposition concerning equilibrium indeterminacy, we define the following three critical values for the externality parameter:

$$\theta^a \equiv \frac{\psi}{\alpha(1+\chi)} - 1,$$

$$\theta^b \equiv \frac{M + s^*}{M(1+B)} - 1 > 0,$$

$$\theta^c \equiv \frac{\delta s^* - \rho M}{\Sigma} - 1,$$

where $\Sigma \equiv \delta\alpha(1+\chi)(M + s^*)/\psi - \rho M(1+B).$

Proposition 2. *The equilibrium exhibits indeterminacy if either*

(i). $\Sigma > 0 \;\; and \;\; \theta \in \left(\theta^b, \theta^a\right),$

(ii). $\Sigma < 0 \;\; and \;\; either \;\; \theta \in \left(\max\left\{\theta^a, \theta^c\right\}, \theta^b\right) \;\; or \;\; \theta \in \left(\theta^b, \min\left\{\theta^a, \theta^c\right\}\right).$

Proof: See Appendix C.

Proposition 2 shows that indeterminacy arises when the intensity of the externalities in the investment sector θ is within certain ranges. In particular, a sufficiently large intensity of the externality in the investment sector is necessary. As in Harrison (2001), where the possibility of indeterminacy is studied when preferences are separable between consumption and leisure, indeterminacy arises when households expect the returns from capital to increase when they invest in capital. Obviously, this requires a minimum value of the investment externality. Following Guo and Harrison (2010), we refer to this minimum value by θ^{min}.

The following result shows that, by introducing an empirically plausible assumption, θ^{min} is always equal to θ^{b}. [7]

Proposition 3. Assume that $\delta>\rho$. Then $\theta^{min}=\theta^{b}$ *and the equilibrium exhibits indeterminacy if either*

(i). $\Sigma > 0 \, and \ \theta \in \left(\theta^b, \theta^a\right)$,

(ii). $\Sigma < 0 \, and \ \left(\theta^b, \min\left\{\theta^a, \theta^c\right\}\right)$.

Proof: See Appendix C.

In the following section, we analyze the determinants of this minimum intensity of the externality necessary to have equilibrium indeterminacy. We focus our analysis to study the determinants of θ^{min}, as the literature has already shown that the main difficulty to explain equilibrium indeterminacy is to justify a sufficiently large intensity of this externality.[8]

4. Analysis of equilibrium indeterminacy

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7 Guo and Harrison (2010) assume that ρ=0.01 and δ=0.025. These are standard values of these parameters that clearly satisfy that δ>ρ, which is the assumption introduced in Proposition 3.

8 Basu and Fernald (1997) show that increasing returns to scale are around 0.33 in the durable goods sector of the US economy. This introduces a clear limit on the plausible values of $\theta^{\min}$.

This section envisages how the income effect on the labor supply and the elasticity of the labor supply affect the minimum intensity of the externalities needed for indeterminacy. As follows from Proposition 3, this minimum value is determined by θ^b, $\theta^{\min} = \theta^b$, which is rewritten as follows:

$$\theta^{\min}(\phi,\chi) = \frac{M + s^*}{M(1+B)} - 1 = \frac{1+\chi-\phi(1-\alpha)}{[(1+\chi)(1-s^*)-\phi(1-\alpha)]\left[1+\frac{(1-\phi)(1-\alpha)s^*}{(1-s^*)[1+\chi-\phi(1-\alpha)]}\right]} - 1. \tag{13}$$

From using this expression, we obtain the following result:

Proposition 4. *Assume that* δ>ρ. *Then*

(i) *a decrease in* ϕ *lowers* $\theta^{\min}$,

(ii) *a larger* χ *reduces* $\theta^{\min}$ *if* ϕ *is sufficiently large, but increases* $\theta^{\min}$ *if* ϕ *is sufficiently small.*

Part (i) is easily proved by differentiating (13) with respect to ϕ:

$$\frac{d\theta^{\min}}{d\phi} = \frac{(1-\alpha)s^*}{M(1+B)(1+\chi)}\left\{\frac{1}{M} + \frac{(\chi+\alpha)}{(1-s^*)(1+B)[1+\chi-\phi(1-\alpha)]}\right\} > 0.$$

To prove part (ii), we differentiate (13) with respect to χ:

$$\frac{d\theta^{\min}}{d\chi} = \frac{(1-\alpha)s^*}{M(1+B)(1+\chi)}\left\{\frac{1-\phi}{(1-s^*)(1+B)[1+\chi-\phi(1-\alpha)]} - \frac{\phi}{M(1+\chi)}\right\}. \tag{14}$$

As follows from (14), if ϕ=1 and, thus, there are no income effect on the labor supply, the derivative is unambiguously negative, implying that a higher χ lowers the minimum value of the investment externality needed for indeterminacy. In contrast, if ϕ<1 and, thus, there is an income effect on the labor supply, the derivative (14) can be positive and indeed is positive for a sufficiently small value of ϕ. In this case, a larger value of χ increases the minimum value of the investment externality needed for indeterminacy.

To understand part (i), we follow Guo and Harrison (2010) and write down the discrete-time version of the consumption Euler equation, which equates the MRS with the MPK, as follows:[9]

$$\frac{\frac{c_{t+1}}{\phi} - \beta \frac{L_{t+1}^{1+\chi} c_{t+1}^{1-\phi}}{1+\chi}}{\frac{c_t}{\phi} - \beta \frac{L_t^{1+\chi} c_t^{1-\phi}}{1+\chi}} = \frac{1}{1+\rho}\left(\frac{r_{t+1} + (1-\delta) p_{t+1}}{p_t}\right). \tag{15}$$

Note that, when ϕ=1, (15) reduces to the corresponding equation in Guo and Harrison (2010, p. 292). Indeterminacy arises because optimistic expectations on the return to capital cause agents to move productive resources out of the consumption sector and into the investment sector. This obviously reduces current consumption and increases both future consumption and future employment. Moreover, as explained by Benhabib and Farmer (1996) and Harrison (2001), due to increasing returns to scale, the increase in investment reduces the price in period *t* and increases the price in period *t*+1. These changes modify the Euler equation. First, postponing consumption increases the MRS. This increase is limited by the increase in L_{t+1}. Second, the MPK also increases due to

9 See Appendix D for the derivation of equation (15).

the evolution of relative prices. The equilibrium exhibits indeterminacy when optimistic expectations are fulfilled in equilibrium, which requires that the Euler condition holds in equality. As the MPK increases, the MRS must increase sufficiently to maintain the equality in the Euler condition. This can only happen if the increase in employment is not too large, which requires either a sufficiently small LSE (large χ) or a sufficiently large investment externality (large θ). On the one hand, postponing consumption causes the increase in capital and wages in period t+1. This causes a limited increase in the labor supply when the LSE is small. On the other hand, as θ is larger, L_{t+1} increases by less, as the same increase in the households income can be achieved with a smaller rise in L_{t+1} when θ is larger.

When there is an income effect (ϕ<1), indeterminacy is explained by the effect of a falling c_t and a rising c_{t+1} on the Euler equation displayed in (15). Postponing consumption has two opposing effects on the MRS in (15). First, as in the situation with ϕ=1, there is a linear effect, which increases the MRS. Second, due to ϕ<1, there is an additional effect, which decreases the MRS. Thus, the presence of the income effect on the labor supply limits the increase in the MRS due to falling c_t and rising c_{t+1}. It follows that, to maintain the equality in (15), the required increase in the MPK is smaller. Therefore, a smaller minimum value of the investment externality is needed for indeterminacy.

To understand part (ii), note that the additional negative effect of postponing consumption on the MRS in (15) is small when ϕ is large and, thus, the income effect on the labor supply is small. Then, as in the case with ϕ=1, when ϕ is sufficiently large, increases in both χ and θ lead to smaller changes in L_{t+1}. Hence, in this case, the higher χ (the lower LSE), the lower the increasing returns needed to keep the increase in L_{t+1} small enough to satisfy (15).

However, when ϕ is sufficiently small and, thus, the income effect is large, the additional negative effect of postponing consumption on the MRS is large. Then, the mechanism completely changes as the negative effects are augmented by the increase in L_{t+1}. This explains that $\theta^{\min}$ increases with χ. To illustrate this, we take as an example the limiting case with ϕ=0. In this case, this additional negative effect of falling c_t and rising c_{t+1} on the MRS becomes linear. Indeed, now $\Delta = \log c - \beta \frac{L^{1+\chi}}{1+\chi}$ and the utility is $\log(\log c - \beta \frac{L^{1+\chi}}{1+\chi})$, which is otherwise the same as a KPR utility (King, Plosser and Rebelo, 1988) except for a logarithmic function over the KPR utility. Then, (15) reduces to:

$$\left(\frac{c_{t+1}}{c_t}\right)\left(\frac{\log c_{t+1} - \beta \frac{L_{t+1}^{1+\chi}}{1+\chi}}{\log c_t - \beta \frac{L_t^{1+\chi}}{1+\chi}}\right) = \frac{1}{1+\rho}\left\{\frac{r_{t+1} + (1-\delta) p_{t+1}}{p_t}\right\}, \tag{16}$$

in which the term in the first brackets on the left-hand side is the same as the left-hand side of the Euler equation in Harrison (2001, p. 754). Hence, the mechanism explaining the effect of the LSE is reminiscent of Harrison (2001): as LSE decreases, labor is less easily drawn out of leisure. Now, decreases in χ and increases in θ both lead to smaller changes in L_{t+1}. Thus, in this case, the higher χ (the lower LSE), the higher the degree of increasing returns needed to keep the increase in L_{t+1} small enough to satisfy (16). The same reasoning goes for the situation when ϕ is sufficiently small.

In what follows, we solve numerically the model to gain some insight on the plausible values of $\theta^{\min}$. To this end, we set the values of different parameters as in Guo and Harrison (2010). They use the following baseline parameter values in a quarterly frequency to quantify the magnitude of the investment externality necessary for indeterminacy: α=0.3, ρ=0.01, δ=0.025 and χ=15. These parameter values imply a labor income share equal to 70%, a long-run savings rate of 21% and a value of the LSE equal to 6.6%.[10]

To illustrate how the minimum value of the intensity of the investment externality needed for indeterminacy changes for Σ<0 and Σ>0, we consider two different values of ϕ. We first assume that ϕ=0.5, which implies that Σ<0, $\theta^a = 2.26$, $\theta^b = 0.27$ and $\theta^c = 11.33$. As Σ<0, according to Proposition 3, if $\theta \in (\theta^b, \max\{\theta^a, \theta^c\}) = (0.27,\ 2.26)$, the equilibrium exhibits indeterminacy. Next, we assume that ϕ=0.9, which implies that Σ<0, $\theta^a = 2.20$, $\theta^b = 0.28$ and $\theta^c =$ -78.04. As Σ>0, according to Proposition 3, indeterminacy emerges if $\theta \in (\theta^b, \theta^a) =$ (0.28, 2.20). We should note that when ϕ is smaller, not only the required minimum investment externality is smaller, but also the range of this

10 The value of the LSE considered in Guo and Harrison (2010) is a conservative value. Other authors consider values of this elasticity larger and close to 0.5.

externality for which the equilibrium exhibits indeterminacy is larger.[11] This clearly shows that the possibility of indeterminacy increases with the intensity of the income effects.

There is not a consensus in the literature on the value of the intensity of the income effect. Therefore, to obtain an inside on the possible values of the minimum externality needed for indeterminacy, Figure 1 shows the value of $\theta^{\min}$ for different values of ϕ using the baseline parameter values. Figure 1 illustrates the positive relationship between the value of ϕ and the minimum value of the investment externality $\theta^{\min}$ needed for indeterminacy. Figure 2 illustrates the relationship between χ and $\theta^{\min}$ for different values of ϕ. When the value of ϕ is large, the relationship between χ and $\theta^{\min}$ is negative, as explained in Proposition 4. By contrast, when the value of ϕ is small, the relationship between χ and $\theta^{\min}$ is positive.

We must note that in a model with a non-separable preference, Dufourt, et al. (2017) have also shown that local indeterminacy occurs in two-sector models under plausible configurations of the parameters. They show that for any given size of the income effect there is a non-empty range of values for the LSE and the IES in consumption such that indeterminacy occurs. In contrast, our analysis

11 The range of the externality for which the equilibrium exhibits indeterminacy is (0.29, 2.18) when ϕ=1 and, hence, there are no income effects. Note that this range is clearly smaller than the range of values obtained when income effects are introduced.

outlines that indeterminacy arises under plausible increasing returns to scale. In particular, in Figures 1 and 2, the increasing returns to scale in the investment sector needed for indeterminacy are smaller than those estimated for the US economy by Basu and Fernald (1997). Moreover, in comparison to their paper, the result illustrated in Figure 2 is new, as they did not show that the relationship between the LSE and the possibility of indeterminacy depends on the intensity of the income effect.

As a final remark, we should mention that in previous versions of the paper we also introduced sector specific externalities in the consumption sector.[12] We obtain numerically the same findings and, hence, we can conclude that our results are robust to the introduction of externalities in the consumption sector.

5. Concluding Remarks

We study an infinite-horizon two-sector model with sector-specific productive externalities and non-separable preferences. We investigate how the income effect on the labor supply and the wage elasticity of the labor supply affect macroeconomic instability or equilibrium indeterminacy. From this analysis, we obtain two main results. First, an increase in the income effect on the labor supply always increases the possibility of indeterminacy. Second, a larger labor supply

12 Previous versions of the paper with sector specific externalities in both sectors are available upon request.

elasticity may increase or decrease the possibility of indeterminacy, depending on the intensity of the income effect.

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Appendix

A. Derivation of the first-order conditions in the household problem

The representative consumer maximizes the discounted sum of utilities, $\int_0^\infty e^{-\rho t} u(C_t, L_t)$, subject to (1a), (1b) and (2). Substituting (1b) into the utility and (1a) into (2), the household solves the following Hamiltonian:

$$\underset{\{L_t, C_t, K_{t+1}\}}{\Re} = \log\left(\frac{1}{\phi}[A_2((1-s)k)^{\alpha}((1-u)L)^{1-\alpha}]^{\phi} - \beta\frac{L^{1+\chi}}{1+\chi}\right) + \lambda\left[\hat{A}_1(sk)^{\alpha}(uL)^{1-\alpha} - \delta k\right]$$

where λ is the co-state variable. The first order conditions with respect to *L, s, u and k,* are, respectively, equations (4a)-(4d) in the main text.

B. Taylor's first-order expansion

Using (10), we rewrite (11) and (12) as:

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - \alpha A_1\left(\frac{(1-\alpha)A_2^{\phi}}{\beta}\right)^{\frac{(1-\alpha)(1+\theta)}{\psi}} k^{\varepsilon-1}s^{\theta}(1-s)^{\frac{(1-\alpha)(1+\theta)(\phi-1)}{\psi}},$$

$$\frac{\dot{k}}{k} = A_1\left(\frac{(1-\alpha)A_2^{\phi}}{\beta}\right)^{\frac{(1-\alpha)(1+\theta)}{\psi}} k^{\varepsilon-1}s^{1+\theta}(1-s)^{\frac{(1-\alpha)(1+\theta)(\phi-1)}{\psi}} - \delta.$$

We define Λ=ln(λ), K=ln(k) and S=ln(s) to rewrite the previous system as:

$$\dot{\Lambda} = \rho + \delta - \alpha A_1\left(\frac{(1-\alpha)A_2^{\phi}}{\beta}\right)^{\frac{(1-\alpha)(1+\theta)}{\psi}} e^{(\varepsilon-1)K+\theta S}(1-e^S)^{\frac{(1-\alpha)(1+\theta)(\phi-1)}{\psi}},$$

$$\dot{K} = A_1\left(\frac{(1-\alpha)A_2^{\phi}}{\beta}\right)^{\frac{(1-\alpha)(1+\theta)}{\psi}} e^{(\varepsilon-1)K+(1+\theta)S}(1-e^S)^{\frac{(1-\alpha)(1+\theta)(\phi-1)}{\psi}} - \delta.$$

From Taylor's first-order expansion around the steady state, we obtain:

$$\begin{pmatrix} \dot{\Lambda} \\ \dot{K} \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{\Lambda}}{\partial \Lambda} & \frac{\partial \dot{\Lambda}}{\partial K} \\ \frac{\partial \dot{K}}{\partial \Lambda} & \frac{\partial \dot{K}}{\partial K} \end{pmatrix} \begin{pmatrix} \Lambda - \Lambda^* \\ K - K^* \end{pmatrix}.$$

Using (10), we obtain:

$$\frac{\partial S}{\partial \Lambda} = \frac{\lambda^*}{s^*}\frac{\partial s}{\partial \lambda} = -\left[1 - s^* - \frac{(1-\alpha)\phi}{1+\chi}\right]\frac{\psi(1-s^*)}{\Omega},$$

and

$$\frac{\partial S}{\partial K} = \frac{k^*}{s^*}\frac{\partial s}{\partial k} = \varepsilon \frac{\partial S}{\partial \Lambda}.$$

We use these equations to obtain the value of the elements in the Jacobian matrix evaluated at the steady state as follows:

$$\begin{aligned} \frac{\partial \dot{\Lambda}}{\partial \Lambda} &= -(\rho+\delta)\frac{\partial S}{\partial \Lambda}\left[\frac{(1+\theta)(1-\phi)(1-\alpha)\exp(S)}{\psi[1-\exp(S)]} + \theta\right] \\ &= -(\rho+\delta)\frac{\partial S}{\partial \Lambda}\left[\frac{(1+\theta)(1-\phi)(1-\alpha)s^*}{\psi(1-s^*)} + \theta\right], \end{aligned}$$

$$\frac{\partial \dot{\Lambda}}{\partial K} = \varepsilon\left(\frac{\partial \dot{\Lambda}}{\partial \Lambda} - (\rho+\delta)\right) + (\rho+\delta),$$

$$\frac{\partial \dot{K}}{\partial \Lambda} = \delta\left[\frac{1}{\varepsilon}\left(\frac{\partial S}{\partial K} - \frac{1}{\rho+\delta}\frac{\partial \dot{\Lambda}}{\partial K} + 1\right) - 1\right],$$

$$\frac{\partial \dot{K}}{\partial K} = \delta\left(\frac{\partial S}{\partial K} - \frac{1}{\rho+\delta}\frac{\partial \dot{\Lambda}}{\partial K}\right) = \varepsilon\left(\delta + \frac{\partial \dot{K}}{\partial \Lambda}\right) - \delta.$$

The determinant and trace of the Jacobian matrix J are, respectively:

$$Det(J) = \delta\left(\frac{\partial \dot{\Lambda}}{\partial \Lambda}\frac{\partial S}{\partial K} - \frac{\partial \dot{\Lambda}}{\partial K}\frac{\partial S}{\partial \Lambda}\right) = -\delta(\rho+\delta)(1-\varepsilon)\frac{\partial S}{\partial \Lambda} = \frac{\delta\xi\psi(1-s^*)}{1+\chi}\frac{(1-\varepsilon)}{\Omega},$$

$$Tr(J) = \frac{\partial S}{\partial \Lambda}\left\{\delta\varepsilon - (\rho+\delta)\left[\frac{(1+\theta)(1-\phi)(1-\alpha)s^*}{\psi(1-s^*)} + \theta\right]\left(1 - \frac{\delta\varepsilon}{\rho+\delta}\right)\right\} - \delta(1-\varepsilon)$$

$$= -M\,\frac{\psi(1-s^*)}{\Omega}\left\{\delta\varepsilon - (\rho+\delta)\left[(1+\theta)B + \theta\right]\left(1 - \frac{\delta\varepsilon}{\rho+\delta}\right)\right\} - \delta(1-\varepsilon),$$

where ε, ξ, M and B are defined in the main text.

C. Proof of Propositions 2 and 3

Since the two-dimensional dynamical system includes a predetermined variable and a jump variable, equilibrium indeterminacy occurs if the Jacobian matrix has two roots with negative real parts. This occurs when *Det*(*J*)>0 and *Tr*(*J*)<0. To analyze these conditions, we start by studying the conditions for *Det*(*J*)>0, followed by the conditions for *Tr*(*J*)<0.

First, *Det*(*J*)>0 when (1–*ε*)/Ω>0. Two cases arise: (i) *ε*>1 and Ω<0; and (ii) *ε*<1 and Ω>0. From the definitions of *ε* and Ω, the following two lemmas can directly be proven:

Lemma C1. $\varepsilon > (<) 1$ *if and only if* $\theta > (<) \theta^a$, where $\theta^a \equiv \frac{\psi}{\alpha(1+\chi)} - 1 > 0.$

Lemma C2. $\Omega > (<) 0$ *if and only if* $\theta > (<) \theta^b$, where $\theta^b \equiv \frac{M+s^*}{M(1+B)} - 1 > 0.$

These two lemmas imply that *Det*(*J*)>0 if either $\theta \in (\theta^a, \theta^b)$ when Ω<0, or $\theta \in (\theta^b, \theta^a)$ when Ω>0.

Next, we analyze the conditions for $Tr(J)<0$. Using the expressions of $Tr(J)$ and Ω, we rewrite

$$\Omega\cdot Tr\left(J\right)=\psi(1-s^{*})\left[\delta s^{*}-\rho M-(1+\theta)\Sigma\right],$$

where $\Sigma\equiv\delta(M+s^{*})\alpha(1+\chi)/\psi-\rho M(1+B)$. Thus, Tr(J)<0 if (i) $\Omega Tr(J)>0$ and Ω<0 or (ii) $\Omega Tr(J)<0$ and Ω>0.

The following lemma follows directly from the definitions of Σ, *M*, and *B.*

Lemma C3. Let $\theta^{c}\equiv\frac{\delta s^{*}-\rho M}{\Sigma}-1$. If Σ>0 and $\theta<(>)\theta^{c}$, then $\Omega Tr(J)>(<)0$, and if Σ<0 and $\theta>(<)\theta^{c}$, then $\Omega Tr(J)>(<)0$.

Lemmas C1 to C3 imply that both roots have negative real parts if (i) $\theta\in\left(\theta^{a},\min\left\{\theta^{b},\theta^{c}\right\}\right)$ or $\theta\in\left(\max\left\{\theta^{b},\theta^{c}\right\},\theta^{a}\right)$ when $\Sigma>0$ and (ii) $\theta\in\left(\max\left\{\theta^{a},\theta^{c}\right\},\theta^{b}\right)$ or $\theta\in\left(\theta^{b},\min\left\{\theta^{a},\theta^{c}\right\}\right)$ when $\Sigma<0.$

The previous conditions can be further simplified by using the following lemma that directly follows from the expression of θ^{c}.

Lemma C4. $\theta^{c}<0$ if and only if Σ>0.

If Σ>0, then $\theta^{a}>0,\theta^{b}>0$ and $\theta^{c}<0$. Therefore, when Σ>0, the interval of θ for which the equilibrium exhibits indeterminacy is $\left(\theta^{b},\theta^{a}\right)$. This proves Proposition 2.

In what follows, we assume that δ>ρ and Σ<0 in order to prove Proposition 3. First, Σ<0 implies that

$$\frac{\psi M(1+B)}{(M+s^*)\alpha(1+\chi)} > \frac{\delta}{\rho} > 1$$

This inequality directly implies that $\theta^a > \theta^b$ and, hence, the intervals in Proposition 2 simplify to those of Proposition 3. This proves Proposition 3.

D. Derivation of equation (15)

The household maximizes $\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \log\left(\frac{c_t^{\phi}}{\phi} - \beta\frac{L_t^{1+\chi}}{1+\chi}\right)$, subject to the following budget constraint: $r_t k_t + w_t L_t = c_t + p_t[k_{t+1} - (1-\delta)k_t]$, where w_t is the wage rate. Denote by λ_t the shadow price of the budget constraint. Then, the first order conditions with respect to c_t and k_{t+1} give, respectively:

$$\frac{1}{\frac{c_t^{\phi}}{\phi} - \beta\frac{L_t^{1+\chi}}{1+\chi}} c_t^{\phi-1} = \lambda_t, \tag{D1}$$

$$\lambda_t p_t = \frac{1}{1+\rho}\lambda_{t+1}[r_{t+1} + (1-\delta)p_{t+1}]. \tag{D2}$$

Substituting (D1) into (D2) yields:

$$\frac{(\frac{c_{t+1}^{\phi}}{\phi} - \beta\frac{L_{t+1}^{1+\chi}}{1+\chi})c_{t+1}^{1-\phi}}{(\frac{c_t^{\phi}}{\phi} - \beta\frac{L_t^{1+\chi}}{1+\chi})c_t^{1-\phi}} = \frac{1}{1+\rho}\left(\frac{r_{t+1} + (1-\delta)p_{t+1}}{p_t}\right). \tag{D3}$$

With manipulation, (D3) leads to (15).

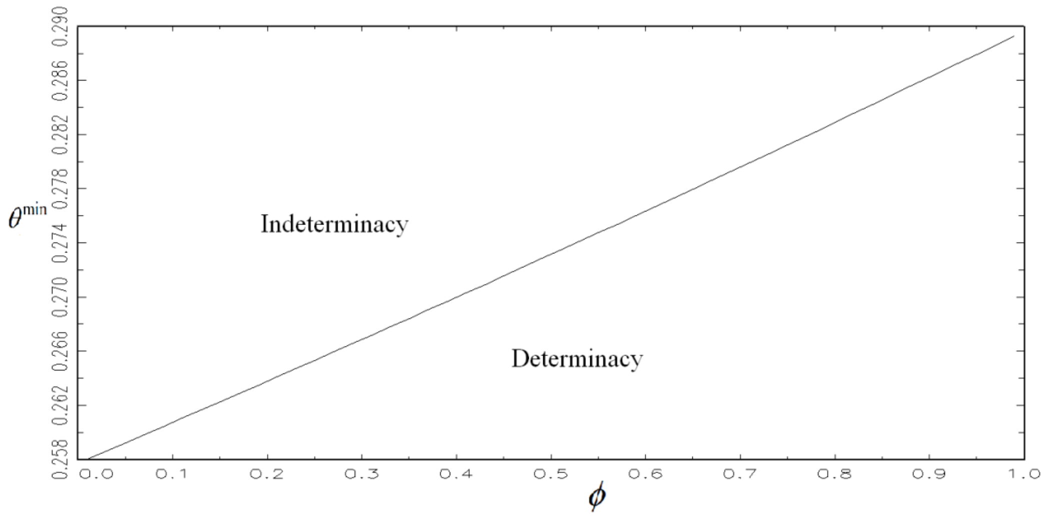

Figure 1. $\theta^{\min}$ **as a function of** $\boldsymbol{\phi}$**.**

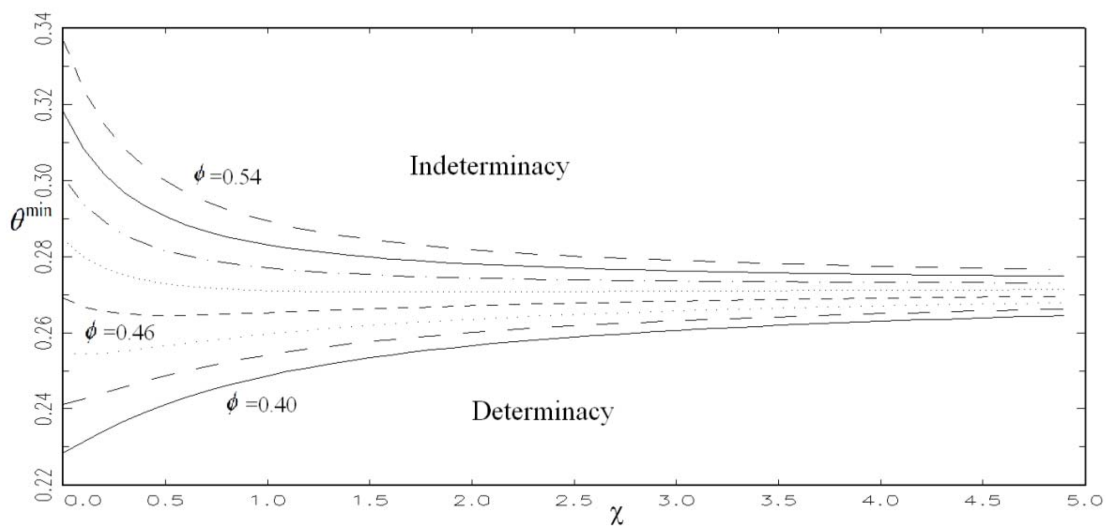

Figure 2. $\theta^{\min}$ **as a function of** χ **for different values of** $\boldsymbol{\phi}$**.**