Measurement of Technical Efficiency in Stochastic Frontier Analysis with Limited and Qualitative Dependent Variable

Shih-Tang Hwu* and Wen-Jen Tsay

August 11, 2017

Abstract

As vividly demonstrated in Maddala (1983), limited and qualitative data have been widely employed in modern econometric analysis. However, analytical methods for evaluating technical efficiency of stochastic frontier analysis can only be applied to continuous dependent variable. This paper provides closed form formulae for evaluating the technical efficiency of stochastic frontier analysis with limited and qualitative dependent variable. Monte Carlo experiments reveal that the finite sample performances of our formulae are promising.

Key words: Stochastic frontier analysis; Limited dependent variable; Technical efficiency

*We thank for the valuable comments from the DEAIC2017 conference at Hefei, China. Corresponding to Hwu: Department of Economics, University of Washington. Tsay: The Institute of Economics, Academia Sinica. 128 Academia Road, Sec. 2, Taipei, Taiwan.

1 Introduction

This paper proposes analytical formulae for evaluating the technical efficiency (TE, hereafter) of stochastic frontier analysis (SFA, hereafter) with limited and qualitative dependent variable (LSFA, hereafter). As pointed out by Kumbhakar (1990), measuring TE of individual observation is one of the main focus of SFA. In the large and growing SFA literature, the formula proposed by Jondrow, Lovell, Materov, and Schmidt (1982) (JLMS, hereafter) and Battese and Coelli (1988) (BC, hereafter) for evaluating TE are widely used in empirical applications. However, these two formulae are only applicable to the models with continuous dependent variable as proposed by Aigner, Lovell, and Schmidt (1977)[1]. When the dependent variable contains limited or qualitative observations, it is inappropriate to apply conventional JLMS and BC efficiency estimates without modification. Nevertheless, there exists no analytic method which can evaluat TE analytically under this circumstance. We intend to fill the gap of the literature to generalize JLMS and BC's methods for the SFA with limited or qualitative dependent variables.

We propose two efficiency estimates, JLMS interval efficiency estimate and BC interval efficiency estimate. Apparently, the potential applications of the proposed estimation method is far reaching. First, many of the survey data are interval or qualitative data. In particular, Tsay and Fu (2016) suggest an analytic formula for evaluating the likelihood function of the SFA model with an interval dependent variable when the inefficient term follows a half-normal distribution as considered in ALS (1977). Second, observed wage is usually censored due to the existence of the regulation of minimum wage. Many research have applied SFA framework to analyze the wage frontier model. Following the job search mechanism considered in Hofler and Murphy (1992, 1994), we can model the wage determination process in a stochastic frontier framework,

$$w_i = w_i^r + u_i = x_i'\beta + v_i + u_i,$$

where x_i are the reservation wage (w_i^r) determinants with coefficient β. The two-sided random error v_i denotes statistical noise and the non-negative $u_i \geq 0$ represents the

[1] See Greene (1997) and Kumbhakar and Lovell (2000) for thoroughly literature reviews.

degree by which the worker's observed wage exceeds the reservation wage. Following this job search framework, Tsay, Huang, Fu, and Ho (2013) apply SFA framework to analyze the wage frontier model. The data-generating process (DGP) in Tsay et al. (2013) is:

$$\begin{cases} w_i^* = x_i'\beta + v_i + u_i, \\ w_i = w_i^*, \quad \text{if} \quad w_i^* > w^{min}, \\ w_i = w^{min}, \quad \text{if} \quad w_i^* \leq w^{min}, \end{cases}$$

where w^{min} denotes the minimum wage and serves as the censoring points for the observed wage w_i.

The remaining parts of this paper are arranged as follows: Section 2 presents the proposed formulae for evaluating TE and their properties. Section 3 considers Monte Carlo experiments to show the finite sample performance of the proposed interval efficiency estimates. Section 4 provides the conclusion.

2 Limited and Qualitative Stochastic Frontier Analysis

Consider a conventional SFA regression in logarithm form:

$$y_i = x_i'\beta + \varepsilon_i, \tag{1}$$

where y_i and ε_i are the i^{th} observation of dependent variable and the composite random error, respectively; x_i' is a $1 \times k$ vector of the i^{th} observation on k regressors; and β is a $k \times 1$ vector of unknown parameters to be estimated. We adopt the notation of Greene (2005) and specify the composite random error ε_i as:

$$\begin{aligned} &\varepsilon_i = v_i + S u_i, \\ &v_i \sim N\left(0, \sigma_v^2\right), \\ &u_i \sim N^+\left(0, \sigma_u^2\right), \quad u_i > 0, \end{aligned} \tag{2}$$

where v_i denotes the random noise, u_i denotes the technical efficiency.[2] S is a pre-specified number, which is equal to -1 if the frontier describes production or 1 if the frontier describes cost. v_i and u_i are independent of each other, and are independent of x_i. Aigner et al. (1977) show that, under these distribution assumptions, the probability density function (pdf) of the composite error $\varepsilon_i = v_i + Su_i$ can be denoted as:

$$f_S(\varepsilon_i) = \frac{2}{\sigma}\phi\left(\frac{\varepsilon_i}{\sigma}\right)\Phi\left(S\lambda\frac{\varepsilon_i}{\sigma}\right), \tag{3}$$

where $\sigma^2 = \sigma_u^2 + \sigma_v^2$, $\lambda = \dfrac{\sigma_u}{\sigma_v}$, $\phi(.)$ and $\Phi(.)$ denote the pdf and the cumulative density function (cdf) of standard normal distribution, respectively. This distribution is also known as skew-normal distribution (Azzalini, 1985). For notation conveniency, we define $f_S(.)$ and $F_S(.)$ represent pdf and cdf of skew-normal distribution with the pre-specified S, respectively.

Following the definition in Aigner et al. (1977), we express the TE of observation i in Eq. (1), namely TE_i, as:

$$TE_i = \exp(-u_i). \tag{4}$$

2.1 JLMS interval efficiency estimate

Since the inefficiency term u_i is unobservable, JLMS (1982) suggest that the conditional expectation of u_i given ε_i can be used as an estimator of u_i. Given that u_i follow a half normal distribution, the corresponding JLMS efficiency estimate is:

$$TE_i^{JLMS} = \exp(-\widehat{u}_i), \tag{5}$$

where

$$\widehat{u}_i = E(u_i|\varepsilon_i) = S\frac{\sigma_u^2}{\sigma^2}\varepsilon_i + \frac{\sigma_u\sigma_v}{\sigma}\frac{\phi\left(\lambda\frac{\varepsilon_i}{\sigma}\right)}{\Phi\left(S\lambda\frac{\varepsilon_i}{\sigma}\right)}. \tag{6}$$

[2] There are many candidates for the distribution assumption of the efficiency component. Greene (1990) employed cross-sectional data to estimate a stochastic cost frontier and reported that the estimated TE from four different distribution assumptions, half normal, exponential, truncated normal and gamma, are quite similar. Ritter and Simar (1997) also argue that we should use relatively simple distribution, such as half normal or exponential, rather than truncated normal or gamma. Since the half normal assumption has been used most frequently in the literature (Bauer, 1990), we only consider half normal case here.

The formulae in Eq. (5) and Eq. (6) have been widely applied in empirical work. However, when the dependent variable is limited or qualitative variable, the difficulty arises since ε_i is limited to be within the corresponding interval.

Without loss of generality, we assume the censoring interval of dependent variable is at (A, B),

$$\begin{cases} y_i^* = x_i'\beta + \varepsilon_i, \quad i = 1, 2, \ldots, N; \\ y_i = y_i^*, \quad \text{if } A < y_i^* < B; \\ y_i = A, \quad \text{if } y_i^* \leq A; \\ y_i = B, \quad \text{if } y_i^* \geq B, \end{cases} \tag{7}$$

where y_i^* denotes the actual dependent variable which is only observable when its value fall into the range (A, B).

In the case with either censored or interval data, A and B in Eq. (7) is clearly defined by data. In the case with qualitative data, A and B can be consistently estimated through maximum likelihood estimation.

Given the DGP in Eq. (7), we can directly transform the estimator in Eq. (6) as:

$$\widehat{u}_i = E\left(u_i | A - x_i'\beta < \varepsilon_i < B - x_i'\beta\right) \equiv E\left(u_i | a_i < \varepsilon_i < b_i\right) \tag{8}$$

For convenience, we suppress notation i hereafter. By law of iterated expectation, we express the censored conditional expectation in Eq. (8) as:

$$\begin{aligned} & E\left(u | a < \varepsilon < b\right) \\ &= \frac{1}{F_S\left(\varepsilon\right)_a^b} \int_a^b E\left(u|\varepsilon\right) f_S\left(\varepsilon\right) d\varepsilon \\ &= \frac{1}{F_S\left(\varepsilon\right)_a^b} \int_a^b \left(S\frac{\sigma_u^2}{\sigma^2}\varepsilon + \frac{\sigma_u\sigma_v}{\sigma} \frac{\phi\left(\lambda\frac{\varepsilon}{\sigma}\right)}{\Phi\left(S\lambda\frac{\varepsilon}{\sigma}\right)} \right) \left(\frac{2}{\sigma}\phi\left(\frac{\varepsilon}{\sigma}\right)\Phi\left(S\lambda\frac{\varepsilon}{\sigma}\right)\right) d\varepsilon \\ &= \frac{1}{F_S\left(\varepsilon\right)_a^b} \left[\int_a^b S\frac{\sigma_u^2}{\sigma^2}\varepsilon\phi\left(\frac{\varepsilon}{\sigma}\right)\Phi\left(S\lambda\frac{\varepsilon}{\sigma}\right) d\varepsilon + \int_a^b \frac{2\sigma_u\sigma_v}{\sigma^2}\phi\left(\frac{\varepsilon}{\sigma}\right)\phi\left(\lambda\frac{\varepsilon}{\sigma}\right) d\varepsilon \right] \\ &= E_1 + E_2, \end{aligned} \tag{9}$$

where $F_S\left(\varepsilon\right)_a^b$ denotes $F_S\left(b\right) - F_S\left(a\right)$. It follows that the analysis centers on evaluating

E_1 and E_2 in the above equation.

We first rewrite E_1 in Eq. (9) as:

$$\begin{aligned} E_1 =& \frac{1}{F_S\left(\varepsilon\right)_a^b}\frac{S\sigma_u^2}{\sigma^2}\int_a^b \varepsilon f_{\sigma,\lambda}\left(\varepsilon\right)d\varepsilon \\ =& S\frac{\sigma_u^2}{\sigma^2}E\left(\varepsilon|a<\varepsilon<b\right). \end{aligned} \tag{10}$$

As we can see, the integral part in E_1 is a censored conditional expectation of $f_S\left(\varepsilon\right)$ given that $\varepsilon \in (a,b)$. Using Proposition 1 in Flecher, Allard and Naveau (2009), we can evaluate E_1 as:

$$E_1 = S\frac{\sigma_u^2}{\sigma}\left\{-\sigma\frac{\left[f_S(\varepsilon)\right]_a^b}{\left[F_S(\varepsilon)\right]_a^b}+\frac{2\lambda}{\sqrt{2\pi}\lambda^*}\frac{\left[\Phi\left(\frac{S\lambda^*}{\sigma}\varepsilon\right)\right]_a^b}{\left[F_S(\varepsilon)\right]_a^b}\right\}, \tag{11}$$

where $\lambda^* = \sqrt{1+\lambda^2}$.

We can also recast E_2 in Eq. (9) as:

$$\begin{aligned} E_2 =& \frac{2\sigma_u\sigma_v}{\sigma^2}\frac{1}{F_S\left(\varepsilon\right)_a^b}\int_a^b \phi\left(\frac{\varepsilon}{\sigma}\right)\phi\left(\lambda\frac{\varepsilon}{\sigma}\right)d\varepsilon \\ =& \frac{2\sigma_u\sigma_v}{\sigma^2}\frac{1}{F_S\left(\varepsilon\right)_a^b}\int_a^b \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(\frac{\varepsilon}{\sigma}\right)^2\right]\exp\left[-\frac{\lambda^2}{2}\left(\frac{\varepsilon}{\sigma}\right)^2\right]d\varepsilon \\ =& \frac{\sigma_u\sigma_v}{\pi\sigma^2}\frac{1}{F_S\left(\varepsilon\right)_a^b}\int_a^b \exp\left(-\frac{\lambda^{*2}\varepsilon^2}{2\sigma^2}\right)d\varepsilon. \end{aligned} \tag{12}$$

To evaluate the last term of Eq. (12), we introduce Eq. (7.4.32) of Abramowitz and Stegun (1970):

$$\int \exp\left[-\left(kx^2+2mx+n\right)\right]dx = \frac{1}{2}\sqrt{\frac{\pi}{k}}\exp\left(\frac{m^2-kn}{k}\right)erf\left(\sqrt{k}x+\frac{m}{\sqrt{k}}\right)+C, \tag{13}$$

where $k \neq 0$, C denotes a finite constant, and $erf(.)$ denotes the error function which can be expressed as:

$$erf\left(z\right) = \frac{2}{\sqrt{\pi}}\int_0^z \exp\left(-t^2\right)dt = 2\int_0^{\sqrt{2}z}\phi\left(t\right)dt. \tag{14}$$

Combining Eq. (13) with Eq. (12), we derive E_2 as:

$$
\begin{aligned}
E_2 =& \frac{\sigma_u\sigma_v}{\pi\sigma^2}\frac{1}{F_S(\varepsilon)_a^b}\left[\frac{1}{2}\sqrt{\frac{\pi}{\frac{\lambda^{*2}}{2\sigma^2}}}erf\left(\frac{\lambda^*}{\sqrt{2}\sigma}\varepsilon\right)\right]_a^b \\
=& \frac{\sigma_u\sigma_v}{\sqrt{2\pi}\sigma\lambda^*}\frac{\left[erf\left(\frac{\lambda^*}{\sqrt{2}\sigma}\varepsilon\right)\right]_a^b}{[F_S(\varepsilon)]_a^b}.
\end{aligned} \tag{15}
$$

With the results in Eq.(11) and Eq. (15), we now obtain the JLMS interval efficiency estimate.

Proposition 2.1.1. *Under $v \in N(0, \sigma_v^2)$ and $u \in N^+(0, \sigma_u^2)$, $E(u|a < \varepsilon < b)$ in Eq. (8) can be expressed as:*

$$
\frac{S\frac{\sigma_u^2}{\sigma}\left\{-\sigma\left[f_S(\varepsilon)\right]_a^b + \frac{2\lambda}{\sqrt{2\pi}\lambda^*}\left[\Phi\left(\frac{S\lambda^*}{\sigma}\varepsilon\right)\right]_a^b\right\} + \frac{\sigma_u\sigma_v}{\sqrt{2\pi}\lambda^*\sigma}\left[erf\left(\frac{\lambda^*}{\sqrt{2}\sigma}\varepsilon\right)\right]_a^b}{F_S(\varepsilon)_a^b}. \tag{16}
$$

where $\lambda^* = \sqrt{1+\lambda^2}$.

Proposition 2.1.1 extends the estimation method of JLMS (1982) to the limited and qualitative dependent variable cases. This analytic formula can be easily computed with standard statistic packages, except for the term in the denominator, $F_S(\varepsilon)_a^b$. Nevertheless, this term is easy to compute once we know how to calculate the cdf of skew-normal distribution. Indeed, this problem has been touched upon in Tsay et al. (2013) where they provide an analytical formula which approximate $F_S(\varepsilon_i)$ well. The Monte Carlo experiment conducted in the next section reveals the promising performance of the formula in Proposition 2.1.1.

By construction, the results in Proposition 2.1.1 are identical to those in JLMS (1982) when the censoring interval degenerate to 0. As a corollary, we prove this results in the next Proposition.

Proposition 2.1.2. *Given Proposition 2.1.1, c and ξ are any finite constant,*

$$
\lim_{\xi\to 0} E(u|c < \varepsilon < c+\xi) \to E(u|\varepsilon = c)
$$

where $E\left(u|\varepsilon=c\right)$ *is the formula in JLMS (1982).*

Proof of Proposition 2.1.2 is in the Appendix.

2.2 BC interval efficiency estimate

It is well known in the literature that an alternative estimation method of TE is BC efficiency estimate:

$$TE^{BC}=E\left[\exp\left(-u\right)|\varepsilon\right]=\left[\frac{1-\Phi\left(\frac{\sigma_u\sigma_v-S\lambda\varepsilon}{\sigma}\right)}{\Phi\left(S\lambda\frac{\varepsilon}{\sigma}\right)}\right]\exp\left(-S\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{1}{2}\frac{\sigma_u^2\sigma_v^2}{\sigma^2}\right). \tag{17}$$

This subsection considers the interval BC efficiency estimate when the DGP is Eq. (7),

$$\begin{aligned}
&E\left[\exp\left(-u\right)|a<\varepsilon<b\right]\\
&=\frac{1}{F_S\left(\varepsilon\right)_a^b}\int_a^b E\left[\exp\left(-u\right)|\varepsilon\right]f\left(\varepsilon\right)d\varepsilon\\
&=\frac{2}{\sigma F_S\left(\varepsilon\right)_a^b}\int_a^b\left[\frac{1-\Phi\left(\dfrac{\sigma_u\sigma_v-S\lambda\varepsilon}{\sigma}\right)}{\Phi\left(S\lambda\dfrac{\varepsilon}{\sigma}\right)}\right]\exp\left(-S\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}\right)\phi\left(\frac{\varepsilon}{\sigma}\right)\Phi\left(S\lambda\frac{\varepsilon}{\sigma}\right)d\varepsilon\\
&=\frac{2}{\sqrt{2\pi}\sigma F_S\left(\varepsilon\right)_a^b}\int_a^b\left[1-\Phi\left(\frac{\sigma_u\sigma_v-S\lambda\varepsilon}{\sigma}\right)\right]\exp\left(-S\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}-\frac{\varepsilon^2}{2\sigma^2}\right)d\varepsilon.
\end{aligned} \tag{18}$$

There is no closed form solution for the above integral. Nevertheless, we will show that it can be well approximated with an analytical formula.

Following Eq. (14), it's straightforward to transform $\Phi(x)$ as $\frac{1}{2}+\frac{1}{2}erf\left(\frac{x}{\sqrt{2}}\right)$. Then we can rewrite the last term of Eq. (18) as:

$$\frac{2}{\sqrt{2\pi}\sigma F_S\left(\varepsilon\right)_a^b}\int_a^b\left[\frac{1}{2}-\frac{1}{2}erf\left(\frac{\sigma_u\sigma_v-S\lambda\varepsilon}{\sqrt{2}\sigma}\right)\right]\exp\left(-S\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}-\frac{\varepsilon^2}{2\sigma^2}\right)d\varepsilon \tag{19}$$

Since there exists no analytical form of the erf function, the integral in Eq. (19) cannot be calculated analytically. To solve this problem, we follow the method in Tsay et al. (2013) to approximate the erf function in Eq. (19). Particularly, Tsay et al. (2013) show that, for all $x\geq 0$, $erf(x)$ can be well approximate with a nonlinear function,

$$erf(x)=1-\exp(c_1x+c_2x^2), \tag{20}$$

where $c_1 = -1.0950081470333$ and $c_2 = -0.75651138383854$. Tsay et al. (2013) demonstrate that this approximation has promising finite sample performance. For more detail, see Tsay et al. (2013).

When we apply the approximation method in Eq. (20) to Eq. (19), it requires $\frac{\sigma_u\sigma_v - S\lambda\varepsilon}{\sqrt{2}\sigma}$ in the erf function to be positive. To handle this restriction, we apply the property of the error function, $erf(-x) = -erf(x)$, and discuss the case $S = 1$ and $S = -1$ separately.

Proposition 2.2.1. *Given that* $\tau = \frac{-\sigma_u\sigma_v}{\lambda}$ *and* $S = -1$*, under* $v_i \in N(0, \sigma_v^2)$ *and* $u_i \in N^+(0, \sigma_u^2)$*,* $E(\exp(-u)\,|a < \varepsilon < b)$ *in Eq. (17) can be approximated as:*

Case 1. *If* $\tau < a < b$*,*

$$\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_1}}\frac{1}{F_S(\varepsilon)_a^b}\exp\left(\frac{m_1^2 - k_1n_1}{k_1}\right)\left[erf\left(\sqrt{k_1}b + \frac{m_1}{\sqrt{k_1}}\right) - erf\left(\sqrt{k_1}a + \frac{m_1}{\sqrt{k_1}}\right)\right].$$

Case 2. *If* $a < \tau < b$*,*

$$\frac{1}{\sqrt{2}\sigma}\frac{1}{\sqrt{k_2}}\frac{1}{F_S(\varepsilon)_a^b}\exp\left(\frac{m_2^2 - k_2n_2}{k_2}\right)\left[erf\left(\sqrt{k_2}\tau + \frac{m_2}{\sqrt{k_2}}\right) - erf\left(\sqrt{k_2}a + \frac{m_2}{\sqrt{k_2}}\right)\right]$$
$$-\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_3}}\frac{1}{F_S(\varepsilon)_a^b}\exp\left(\frac{m_3^2 - k_3n_3}{k_3}\right)\left[erf\left(\sqrt{k_3}\tau + \frac{m_3}{\sqrt{k_3}}\right) - erf\left(\sqrt{k_3}a + \frac{m_3}{\sqrt{k_3}}\right)\right]$$
$$+\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_1}}\frac{1}{F_S(\varepsilon)_a^b}\exp\left(\frac{m_1^2 - k_1n_1}{k_1}\right)\left[erf\left(\sqrt{k_1}b + \frac{m_1}{\sqrt{k_1}}\right) - erf\left(\sqrt{k_1}\tau + \frac{m_1}{\sqrt{k_1}}\right)\right].$$

Case 3. *If* $a < b < \tau$*,*

$$\frac{1}{\sqrt{2}\sigma}\frac{1}{\sqrt{k_2}}\frac{1}{F_S(\varepsilon)_a^b}\exp\left(\frac{m_2^2 - k_2n_2}{k_2}\right)\left[erf\left(\sqrt{k_2}b + \frac{m_2}{\sqrt{k_2}}\right) - erf\left(\sqrt{k_2}a + \frac{m_2}{\sqrt{k_2}}\right)\right]$$
$$-\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_3}}\frac{1}{F_S(\varepsilon)_a^b}\exp\left(\frac{m_3^2 - k_3n_3}{k_3}\right)\left[erf\left(\sqrt{k_3}b + \frac{m_3}{\sqrt{k_3}}\right) - erf\left(\sqrt{k_3}a + \frac{m_3}{\sqrt{k_3}}\right)\right],$$

where

$$
\begin{aligned}
k_1 &= \frac{1 - c_2\lambda^2}{2\sigma^2}, \\
m_1 &= -\frac{2\sigma_u^2 + 2c_2\sigma_u^2 + \sqrt{2}c_1\lambda\sigma}{4\sigma^2}, \\
n_1 &= -\frac{\sigma_u^2\sigma_v^2\left(1 + c_2 + \frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}, \\
k_2 &= \frac{1}{2\sigma^2}, \\
m_2 &= -\frac{\sigma_u^2}{2\sigma^2}, \\
n_2 &= -\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}, \\
k_3 &= \frac{1 - c_2\lambda^2}{2\sigma^2}, \\
m_3 &= -\frac{2\sigma_u^2 + 2c_2\sigma_u^2 - \sqrt{2}c_1\lambda\sigma}{4\sigma^2}, \\
n_3 &= -\frac{\sigma_u^2\sigma_v^2\left(1 + c_2 - \frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}.
\end{aligned}
$$

Proposition 2.2.2. *Given that* $\tau = \frac{-\sigma_u\sigma_v}{\lambda}$ *and* $S = 1$*, under* $v_i \in N\left(0, \sigma_v^2\right)$ *and* $u_i \in N^+\left(0, \sigma_u^2\right)$*,* $E\left(\exp\left(-u\right)|a < \varepsilon < b\right)$ *in Eq. (17) can be approximated as:*

Case 1. *If* $-\tau < a < b$*,*

$$
\begin{aligned}
&\frac{1}{\sqrt{2}\sigma}\frac{1}{\sqrt{k_1}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_1^2 - k_1 n_1}{k_1}\right)\left[erf\left(\sqrt{k_1}b + \frac{m_1}{\sqrt{k_1}}\right) - erf\left(\sqrt{k_1}a + \frac{m_1}{\sqrt{k_1}}\right)\right] \\
&- \frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_2}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_2^2 - k_2 n_2}{k_2}\right)\left[erf\left(\sqrt{k_2}b + \frac{m_2}{\sqrt{k_2}}\right) - erf\left(\sqrt{k_2}a + \frac{m_2}{\sqrt{k_2}}\right)\right],
\end{aligned}
$$

Case 2. *If* $a < -\tau < b$*,*

$$
\begin{aligned}
&\frac{1}{\sqrt{2}\sigma}\frac{1}{\sqrt{k_1}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_1^2 - k_1 n_1}{k_1}\right)\left[erf\left(\sqrt{k_1}b + \frac{m_1}{\sqrt{k_1}}\right) - erf\left(-\sqrt{k_1}\tau + \frac{m_1}{\sqrt{k_1}}\right)\right] \\
&- \frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_2}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_2^2 - k_2 n_2}{k_2}\right)\left[erf\left(\sqrt{k_2}b + \frac{m_2}{\sqrt{k_2}}\right) - erf\left(-\sqrt{k_2}\tau + \frac{m_2}{\sqrt{k_2}}\right)\right] \\
&+ \frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_3}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_3^2 - k_3 n_3}{k_3}\right)\left[erf\left(-\sqrt{k_3}\tau + \frac{m_3}{\sqrt{k_3}}\right) - erf\left(\sqrt{k_3}a + \frac{m_3}{\sqrt{k_3}}\right)\right],
\end{aligned}
$$

Case 3. *If* $a < b < -\tau$*,*

$$
\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_3}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_3^2 - k_3 n_3}{k_3}\right)\left[erf\left(\sqrt{k_3}b + \frac{m_3}{\sqrt{k_3}}\right) - erf\left(\sqrt{k_3}a + \frac{m_3}{\sqrt{k_3}}\right)\right],
$$

where

$$
\begin{aligned}
k_1 &= \frac{1}{2\sigma^2}, \\
m_1 &= \frac{\sigma_u^2}{2\sigma^2}, \\
n_1 &= -\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}, \\
k_2 &= \frac{1-c_2\lambda^2}{2\sigma^2}, \\
m_2 &= \frac{2\sigma_u^2+2c_2\sigma_u^2-\sqrt{2}c_1\lambda\sigma}{4\sigma^2}, \\
n_2 &= -\frac{\sigma_u^2\sigma_v^2\left(1+c_2-\frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}, \\
k_3 &= \frac{1-c_2\lambda^2}{2\sigma^2}, \\
m_3 &= \frac{2\sigma_u^2+2c_2\sigma_u^2+\sqrt{2}c_1\lambda\sigma}{4\sigma^2}, \\
n_3 &= -\frac{\sigma_u^2\sigma_v^2\left(1+c_2+\frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}.
\end{aligned}
$$

The derivations of Proposition 2.2.1 and 2.2.2 are in the Appendix.

3 Monte Carlo Experiment

In this section we consider the finite sample performance of the Proposition 2.1.1 and 2.2.1 at different censoring percentiles and true parameters when the DGP is Eq. (7).[3]

Our Monte Carlo experiment design is similar to that of Kumbhakar and Lothgren (1998). The error terms $\varepsilon_i = v_i + Su_i$ are obtained by 10 million random draws of u_i and v_i from $N^+(0, \sigma_u^2)$ and $N(0, \sigma_v^2)$, respectively[4]. We control the variance of two error terms, σ_u and σ_v, and the variance ratio, λ, that reflects the contribution of the variance of u to the total variance of the error term ε. In particular, 3 different variance ratios $\lambda = 0.5, 1, 2$ and 6 censoring percentiles[5] of ε, $(a, b) = (0, 0.01)$,

[3] All programs are written in MATLAB, and are available upon request.

[4] The results of $S = 1$ case are symmetric to what we observe for $S = -1$, so we only report $S = -1$ case here.

[5] We focus on censoring percentiles for expositional purpose, the application of our methods to censoring intervals is straightforward and are available upon request.

$(0.01, 0.05), (0.05, 0.15), (0.85, 095), (0.95, 0.99)$, and $(0.99, 1)$, are considered.

The propositions in last section can estimate the TE of LSFA analytically. As mentioned previously, the cdf of skew-normal distribution $F_S(\varepsilon_i)$ in the denominator of the formula can't be evaluated analytically. To solve this problem, we adopted the approximating method proposed by Tsay and Fu (2016) to approximate the value of the cdf. Tsay and Fu (2016) demonstrate this approximation has promising finite sample performance.

Column 2-4 of Table 1 show the Monte carlo results of Proposition 2.1.1. For various choices of censoring percentiles and true parameters of λ, σ_u and σ_v, the maximum absolute difference ratio between the value calculated by Proposition 2.1.1 and the true value generated by simulation based on the Monte Carlo experiments is about 0.54 percent. However, this value becomes much smaller when the percentiles do not lie at the edge of the distribution. Furthermore, the absolute difference ratio exhibits no apparent pattern either at the censoring percentiles (a, b), or the values of specified parameters λ, σ_u, and σ_v.

Column 5-7 of Table 1 present the results of Proposition 2.2.1. The results are also very promising except when the data is censored at $(0.99, 1)$, which is the extremely censoring point. But since the extremely censoring case is rarely encountered in the empirical studies, we think this problem shouldn't downsize the contribution of this paper too much.

4 Conclusion

This paper is the first to extend the analytic formulae proposed by JLMS (1982) and BC (1988) to evaluate technical efficient in the limited and qualitative stochastic frontier framework, because limited and qualitative data have been widely employed in modern econometric analysis and could be of value to the future studies of the stochastic frontier analysis. The proposed formulae are easy to compute and the Monte Carlo experiments show that the closed form formulae have promising finite sample performance.

5 Appendix

5.1 Proof of Proposition 2.1.2

We can express the limit of proposition 2.1.1 as:

$$
\begin{aligned}
\lim_{\delta\to 0} E\left(u|c<\varepsilon<c+\xi\right) &= \lim_{\delta\to 0} S\frac{\sigma_u^2}{\sigma}\left\{-\sigma\frac{\left[f_S\left(\varepsilon\right)\right]_c^{c+\delta}}{\left[F_S\left(\varepsilon\right)\right]_c^{c+\delta}}+\frac{2\lambda}{\sqrt{2\pi}\lambda^*}\frac{\left[\Phi\left(\frac{S\lambda^*}{\sigma}\varepsilon\right)\right]_c^{c+\delta}}{\left[F_S\left(\varepsilon\right)\right]_c^{c+\delta}}\right\} \\
&\quad+\lim_{\delta\to 0}\frac{\frac{\sigma_u\sigma_v}{\sqrt{2\pi}\sigma\lambda^*}\left[erf\left(\frac{\lambda^*}{\sqrt{2\sigma^2}}\varepsilon\right)\right]_c^{c+\delta}}{\left[F_S\left(\varepsilon\right)\right]_c^{c+\delta}} \\
&= \lim_{\delta\to 0} S\frac{\sigma_u^2}{\sigma}\frac{-\frac{2S\lambda}{\sigma}\phi\left(\frac{c+\delta}{\sigma}\right)\phi\left(\lambda\frac{c+\delta}{\sigma}\right)-\frac{2c}{\sigma^2}\phi\left(\frac{c+\delta}{\sigma}\right)\Phi\left(\lambda\frac{c+\delta}{\sigma}\right)}{f_S\left(c+\delta\right)} \\
&\quad+\lim_{\delta\to 0} S\frac{\sigma_u^2}{\sigma^2}\frac{2S\lambda}{\sqrt{2\pi}}\frac{\phi\left(\lambda^*\frac{c+\delta}{\sigma}\right)}{f_S\left(c+\delta\right)} \\
&\quad+\lim_{\delta\to 0}\frac{2}{\sqrt{2\pi}}\frac{\sigma_u\sigma_v}{\sigma^2}\frac{\phi\left(\lambda^*\frac{c+\delta}{\sigma}\right)}{f_S\left(c+\delta\right)} \\
&= S\frac{\sigma_u^2}{\sigma^2}c-\frac{\frac{2S^2\lambda}{\sigma}\phi\left(\frac{c}{\sigma}\right)\phi\left(\lambda\frac{c}{\sigma}\right)-\frac{2S^2\lambda}{\sqrt{2\pi}\sigma}\phi\left(\lambda^*\frac{c}{\sigma}\right)}{f_S\left(c\right)} \\
&\quad+\frac{2}{\sqrt{2\pi}}\frac{\sigma_u\sigma_v}{\sigma^2}\frac{\phi\left(\lambda^*\frac{c}{\sigma}\right)}{f_S\left(c\right)} \\
&= S\frac{\sigma_u^2}{\sigma^2}c+\frac{\sigma_u\sigma_v}{\sigma}\frac{\phi\left(\lambda\frac{c}{\sigma}\right)}{\Phi\left(S\lambda\frac{c}{\sigma}\right)},
\end{aligned}
$$

where the second equality is based on *L'hospital Rule*, the fourth equality follows from the observation that $\phi\left(\frac{c+\delta}{\sigma}\right)\phi\left(\lambda\frac{c+\delta}{\sigma}\right)=\frac{1}{\sqrt{2\pi}}\phi\left(\lambda^*\frac{c+\delta}{\sigma}\right)$.

5.2 Derivation of Proposition 2.2.1:

Given that $S = -1$ and $\tau = \frac{-\sigma_u\sigma_v}{\lambda}$, we divide the censoring interval into three cases:

Case 1. $\tau < a < b$

In this case, $\frac{\sigma_u\sigma_v+\lambda\varepsilon}{\sqrt{2}\sigma}$ in the erf function in Eq. (19) is always positive in the censoring interval and we can adopt the approximation method in Eq. (20) and rewrite Eq.(19) as:

$$\begin{aligned}
&\frac{2}{\sqrt{2\pi}\sigma F_S(\varepsilon)_a^b}\int_a^b\left\{\frac{1}{2}-\frac{1}{2}\left[1-\exp\left(c_1\left(\frac{\sigma_u\sigma_v+\lambda\varepsilon}{\sqrt{2}\sigma}\right)+c_2\left(\frac{\sigma_u\sigma_v+\lambda\varepsilon}{\sqrt{2}\sigma}\right)^2\right)\right]\right\}\\
&\qquad\times\exp\left(\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}-\frac{\varepsilon^2}{2\sigma^2}\right)d\varepsilon\\
&=\frac{1}{\sqrt{2\pi}\sigma F_S(\varepsilon)_a^b}\int_a^b\exp\left[c_1\left(\frac{\sigma_u\sigma_v+\lambda\varepsilon}{\sqrt{2}\sigma}\right)+c_2\left(\frac{\sigma_u^2\sigma_v^2+\lambda^2\varepsilon^2+2\lambda\sigma_u\sigma_v\varepsilon}{2\sigma^2}\right)\right]\\
&\qquad\times\exp\left(\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}-\frac{\varepsilon^2}{2\sigma^2}\right)d\varepsilon\\
&=\frac{1}{\sqrt{2\pi}\sigma F_S(\varepsilon)_a^b}\int_a^b\exp\left[-\left(\frac{1-c_2\lambda^2}{2\sigma^2}\right)\varepsilon^2\right]\exp\left[-2\left(-\frac{2\sigma_u^2+2c_2\sigma_u^2+\sqrt{2}c_1\lambda\sigma}{4\sigma^2}\right)\varepsilon\right]\\
&\qquad\times\exp\left\{-\left[\frac{-\sigma_u^2\sigma_v^2\left(1+c_2+\frac{c_1\sqrt{2}\sigma}{\sigma_u\sigma_v}\right)}{2\sigma^2}\right]\right\}d\varepsilon.
\end{aligned}\tag{21}$$

Following Eq. (13), we can solve the last term in Eq. (21) as:

$$\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_1}}\frac{1}{F_S(\varepsilon)_a^b}\exp\left(\frac{m_1^2-k_1n_1}{k_1}\right)\left[erf\left(\sqrt{k_1}b+\frac{m_1}{\sqrt{k_1}}\right)-erf\left(\sqrt{k_1}a+\frac{m_1}{\sqrt{k_1}}\right)\right],\tag{22}$$

where

$$\begin{aligned}
k_1&=\frac{1-c_2\lambda^2}{2\sigma^2},\\
m_1&=-\frac{2\sigma_u^2+2c_2\sigma_u^2+\sqrt{2}c_1\lambda\sigma}{4\sigma^2},\\
n_1&=-\frac{\sigma_u^2\sigma_v^2\left(1+c_2+\frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}.
\end{aligned}$$

Case 2. $a < \tau < b$

In this case, $\frac{\sigma_u \sigma_v + \lambda\varepsilon}{\sqrt{2}\sigma}$ is positive when $\tau < \varepsilon$ and negative when $\varepsilon < \tau$. For the $\tau < \varepsilon$ part, we follow the same method in Case 1 to obtain:

$$\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_1}}\frac{1}{F_S(\varepsilon)_a^b}\exp\left(\frac{m_1^2 - k_1 n_1}{k_1}\right)\left[erf\left(\sqrt{k_1}b + \frac{m_1}{\sqrt{k_1}}\right) - erf\left(\sqrt{k_1}\tau + \frac{m_1}{\sqrt{k_1}}\right)\right], \quad (23)$$

where

$$\begin{aligned} k_1 &= \frac{1 - c_2\lambda^2}{2\sigma^2}, \\ m_1 &= -\frac{2\sigma_u^2 + 2c_2\sigma_u^2 + \sqrt{2}c_1\lambda\sigma}{4\sigma^2}, \\ n_1 &= -\frac{\sigma_u^2\sigma_v^2\left(1 + c_2 + \frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}. \end{aligned}$$

For $\varepsilon < \tau$ part, by applying the property of error function, $erf(-x) = -erf(x)$, we can rewrite Eq. (19) as:

$$\frac{2}{\sqrt{2\pi}\sigma F_S(\varepsilon)_a^b}\int_a^b\left[\frac{1}{2} + \frac{1}{2}erf\left(-\frac{\sigma_u\sigma_v + \lambda\varepsilon}{\sqrt{2}\sigma}\right)\right]\exp\left(\frac{\varepsilon\sigma_u^2}{\sigma^2} + \frac{\sigma_u^2\sigma_v^2}{2\sigma^2} - \frac{\varepsilon^2}{2\sigma^2}\right)d\varepsilon. \quad (24)$$

Since $-\frac{\sigma_u\sigma_v + \lambda\varepsilon}{\sqrt{2}\sigma}$ in the erf function of Eq. (24) is positive when $\varepsilon < \tau$, so we can adopt the approximation and recast Eq. (24) as:

$$
\begin{aligned}
& \frac{2}{\sqrt{2\pi}\sigma F_S\left(\varepsilon\right)_a^b}\int_a^b \left\{\frac{1}{2}+\frac{1}{2}\left[1-\exp\left(c_1\left(\frac{-\sigma_u\sigma_v-\lambda\varepsilon}{\sqrt{2}\sigma}\right)+c_2\left(\frac{-\sigma_u\sigma_v-\lambda\varepsilon}{\sqrt{2}\sigma}\right)^2\right)\right]\right\} \\
& \qquad\qquad \times\exp\left(\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}-\frac{\varepsilon^2}{2\sigma^2}\right)d\varepsilon \\
= & \frac{2}{\sqrt{2\pi}\sigma F_S\left(\varepsilon\right)_a^b}\int_a^b \exp\left(\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}-\frac{\varepsilon^2}{2\sigma^2}\right)d\varepsilon \\
& -\frac{1}{\sqrt{2\pi}\sigma F_S\left(\varepsilon\right)_a^b}\int_a^b \exp\left[c_1\left(\frac{-\sigma_u\sigma_v-\lambda\varepsilon}{\sqrt{2}\sigma}\right)+c_2\left(\frac{\sigma_u^2\sigma_v^2+\lambda^2\varepsilon^2+2\lambda\sigma_u\sigma_v\varepsilon}{2\sigma^2}\right)\right] \\
& \qquad\qquad \times\exp\left(\frac{\varepsilon\sigma_u^2}{\sigma^2}+\frac{\sigma_u^2\sigma_v^2}{2\sigma^2}-\frac{\varepsilon^2}{2\sigma^2}\right)d\varepsilon \\
= & \frac{2}{\sqrt{2\pi}\sigma F_S\left(\varepsilon\right)_a^b}\exp\left[-\left(\frac{\varepsilon^2-2\sigma_u^2\varepsilon-\sigma_u^2\sigma_v^2}{2\sigma^2}\right)\right] \\
& -\frac{1}{\sqrt{2\pi}\sigma F_S\left(\varepsilon\right)_a^b}\int_a^b \exp\left[-\left(\frac{1-c_2\lambda^2}{2\sigma^2}\right)\varepsilon^2\right]\exp\left[-2\left(-\frac{2\sigma_u^2+2c_2\sigma_u^2-\sqrt{2}c_1\lambda\sigma}{4\sigma^2}\right)\varepsilon\right] \\
& \qquad\qquad \times\exp\left\{-\left[-\frac{\sigma_u^2\sigma_v^2\left(1+c_2-\frac{c_1\sqrt{2}\sigma}{\sigma_u\sigma_v}\right)}{2\sigma^2}\right]\right\}d\varepsilon.
\end{aligned} \tag{25}
$$

Following Eq. (13), we can solve the last term of Eq. (25) as:

$$
\begin{aligned}
& \frac{1}{\sqrt{2}\sigma}\frac{1}{\sqrt{k_2}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_2^2-k_2n_2}{k_2}\right)\left[erf\left(\sqrt{k_2}\tau+\frac{m_2}{\sqrt{k_2}}\right)-erf\left(\sqrt{k_2}a+\frac{m_2}{\sqrt{k_2}}\right)\right] \\
& -\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_3}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_3^2-k_3n_3}{k_3}\right)\left[erf\left(\sqrt{k_3}\tau+\frac{m_3}{\sqrt{k_3}}\right)-erf\left(\sqrt{k_3}a+\frac{m_3}{\sqrt{k_3}}\right)\right],
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
k_2 &= \frac{1-c_2\lambda^2}{2\sigma^2}, \\
m_2 &= \frac{2\sigma_u^2+2c_2\sigma_u^2-\sqrt{2}c_1\lambda\sigma}{4\sigma^2}, \\
n_2 &= -\frac{\sigma_u^2\sigma_v^2\left(1+c_2+\frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}, \\
k_3 &= \frac{1-c_2\lambda^2}{2\sigma^2}, \\
m_3 &= \frac{2\sigma_u^2+2c_2\sigma_u^2+\sqrt{2}c_1\lambda\sigma}{4\sigma^2}, \\
n_3 &= -\frac{\sigma_u^2\sigma_v^2\left(1+c_2+\frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}.
\end{aligned}
$$

Case 3. $\tau < a < b$

In this case, $\frac{\sigma_u\sigma_v+\lambda\varepsilon}{\sqrt{2}\sigma}$ in the erf function in Eq. (19) is always negative in the censoring interval. Therefore, we can follow the same strategy as the $\varepsilon < \tau$ part in Case 2 to obtain:

$$\begin{aligned}&\frac{1}{\sqrt{2}\sigma}\frac{1}{\sqrt{k_2}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_2^2-k_2n_2}{k_2}\right)\left[erf\left(\sqrt{k_2}b+\frac{m_2}{\sqrt{k_2}}\right)-erf\left(\sqrt{k_2}a+\frac{m_2}{\sqrt{k_2}}\right)\right]\\&-\frac{1}{2\sqrt{2}\sigma}\frac{1}{\sqrt{k_3}}\frac{1}{F_S\left(\varepsilon\right)_a^b}\exp\left(\frac{m_3^2-k_3n_3}{k_3}\right)\left[erf\left(\sqrt{k_3}b+\frac{m_3}{\sqrt{k_3}}\right)-erf\left(\sqrt{k_3}a+\frac{m_3}{\sqrt{k_3}}\right)\right],\end{aligned}\tag{27}$$

where

$$\begin{aligned}k_2&=\frac{1}{2\sigma^2},\\m_2&=-\frac{\sigma_u^2}{2\sigma^2},\\n_2&=-\frac{\sigma_u^2\sigma_v^2}{2\sigma^2},\\k_3&=\frac{1-c_2\lambda^2}{2\sigma^2},\\m_3&=-\frac{2\sigma_u^2+2c_2\sigma_u^2-\sqrt{2}c_1\lambda\sigma}{4\sigma^2},\\n_3&=-\frac{\sigma_u^2\sigma_v^2\left(1+c_2-\frac{\sqrt{2}c_1\sigma}{\sigma_u\sigma v}\right)}{2\sigma^2}.\end{aligned}$$

5.3 Derivation of Proposition 2.2.2:

The derivation of Proposition 2.2.2 is similar to that of Proposition 2.2.1. We omit the details, but the results can be provided upon request.

REFERENCES

1. Abramowitz, M. and I.A. Stegun (1970), *Handbook of Mathematical Functions.* Dover, New York.

2. Aigner, D., C. A. K. Lovell, P. Schmidt (1977), "Formulation and estimation of stochastic frontier production function models," *Journal of Econometrics* 6, 21-37.

3. Azzalini, A. (1985), "A class of distributions which includes the normal ones," *Scandinavian Journal of Statisitics* 12, 171-178.

4. Battese, G. E. and T. J. Coelli (1988), "Prediction of firm-level technical efficiencies with a generalized frontier production function and panel data," *Journal of Econometrics* 38, 387-399.

5. Bauer, P. W. (1990), "Recent developments in the econometric estimation of frontiers," *Journal of Economitrics* 46, 39-56.

6. Flecher, C., D. Allard, and P. Naveau (2009), "Truncated skew-normal distributions: estimation by weighted moments and application to climatic data," *Metron* 68, 331-345.

7. Greene, W. H. (1990), "A Gamma-distributed stochastic frontier model," *Journal of Econometrics* 46, 141-163.

8. Greene, W. H. (1997), "Frontier production functions," in Handbook of Applied Econometrics, Vol. 2, Microeconomics, H. Pesaran and P. Schmidt, eds., Oxford University Press, Oxford.

9. Greene, W. H. (2003), "Simulated likelihood estimation of the normal-gamma stochastic frontier function," *Journal of Productivity Analysis* 19, 179-190.

10. Greene, W. H. (2005), "Fixed and Random Effects in Stochastic Frontier Models," *Journal of Productivity Analysis* 23, 7-32.

11. Greene, W. H. (2010), "A stochastic frontier model with correction for sample selection," *Journal of Productivity Analysis* 34, 15-24.

12. Hofler, R. A., and K. J. Murphy (1992), "Underpaid and overworked: measuring the effects of imperfect information on wages," *Economic Inquiry* 30, 511-29.

13. Hofler, R. A., and K. J. Murphy (1994), "Estimating reservation wages of workers using a stochastic frontier," *Southern Economic Journal* 60, 961-76.

14. Jondrow, J., C. A. K. Lovell, I. S. Materov, and P. Schmidt (1982), "On the estimation of technical inefficiency in the stochastic frontier production function model," *Journal of Econometrics* 19, 233-238.

15. Kumbhakar, S. C. (1990), "Production frontiers and panel data, and time varying technical inefficiency," *Journal of Econometrics* 46, 201-211.

16. Kumbhakar, S. C., and M. Lothgren (1998), "A monte carlo analysis of technical inefficiency predictors," Working paper series in Economics and Finance, No. 229, Stochholm School of Economics.

17. Kumbhakar, S. C. and C. A. K. Lovell (2000), *Stochastic Frontier Analysis,* Cambridge University Press, Cambridge, UK.

18. Maddala, G. S. (1983), *Limited-Dependent and Qualitative Variables in Econometrics,* Cambridge University Press.

19. Ritter, C. and L. Simar (1997), "Pitfalls of normal-gamma stochastic frontier models," *Journal of Productivity Analysis* 8, 167-182.

20. Tsay, W.J., C.J. Huang, T.-T. Fu, and I.-L. Ho (2013), "A simple closed-form approximation for the cumulative distribution function of the composite error of stochastic frontier models," *Journal of Productivity Analysis* 39, 259-269.

21. Tsay, W.J., and T.-T. Fu (2016), Maximum likelihood estimation of stochastic frontier models with interval dependent variables.
Available at SSRN: http://dx.doi.org/10.2139/ssrn.2850827

Table 1. Monte Carlo results for Proposition 2.1.1 and 2.2.1
at various censoring percentiles and true parameters

	JLMS			BC		
Percentiles	P_1	$True_1$	$\frac{\lvert P_1-True_1\rvert}{True_1}$	P_2	$True_2$	$\frac{\lvert P_2-True_2\rvert}{True_2}$
	$\lambda = 1.50 \quad (\sigma_v = 0.50,\ \sigma_u = 0.75,\ \varepsilon_i = v_i - u_i)$					
$(0.00, 0.01)$	0.1645	0.1644	0.0006	0.1821	0.1820	0.0005
$(0.01, 0.05)$	0.2532	0.2531	0.0004	0.2773	0.2772	0.0004
$(0.15, 0.45)$	0.4936	0.4938	0.0004	0.5291	0.5290	0.0002
$(0.55, 0.85)$	0.6728	0.6729	0.0002	0.6990	0.6984	0.0009
$(0.95, 0.99)$	0.8024	0.8024	0.0000	0.8189	0.8155	0.0041
$(0.99, 1.00)$	0.8440	0.8428	0.0014	0.8801	0.8521	0.0328
	$\lambda = 1.00 \quad (\sigma_v = 1.00,\ \sigma_u = 1.00,\ \varepsilon_i = v_i - u_i)$					
$(0.00, 0.01)$	0.1287	0.1294	0.0054	0.1667	0.1674	0.0041
$(0.01, 0.05)$	0.2058	0.2061	0.0015	0.2579	0.2577	0.0008
$(0.15, 0.45)$	0.4011	0.4006	0.0012	0.4621	0.4626	0.0011
$(0.55, 0.85)$	0.5509	0.5509	0.0000	0.6005	0.6004	0.0002
$(0.95, 0.99)$	0.6847	0.6850	0.0004	0.7259	0.7165	0.0131
$(0.99, 1.00)$	0.7361	0.7350	0.0015	0.8003	0.7662	0.0445
	$\lambda = 0.50 \quad (\sigma_v = 2.00,\ \sigma_u = 1.00,\ \varepsilon_i = v_i - u_i)$					
$(0.00, 0.01)$	0.2407	0.2407	0.0000	0.3155	0.3161	0.0019
$(0.01, 0.05)$	0.3026	0.3031	0.0016	0.3785	0.3803	0.0047
$(0.15, 0.45)$	0.4167	0.4168	0.0002	0.4901	0.4889	0.0025
$(0.55, 0.85)$	0.5015	0.5015	0.0000	0.5608	0.5639	0.0055
$(0.95, 0.99)$	0.5902	0.5909	0.0012	0.6439	0.6403	0.0056
$(0.99, 1.00)$	0.6296	0.6318	0.0035	0.6946	0.6745	0.0297

Notes: P_1 and P_2 are computed based on Proposition 2.1.1 and 2.2.1, respectively. $True_1$ and $True_2$ are simulated from the Accept-Reject algorithm based on 10 million independent draws of the distribution $\exp\left(E\left(u|\varepsilon\right)\right)$ and $E\left(\exp\left(u\right)|\varepsilon\right)$, respectively.