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RESEARCH ARTICLE

Two-sided micro-matching with technical progress

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Abstract We develop a two-sided micro-matching framework with heterogeneous workers and machines that permits a complete analysis of technical progress commonly used in Neoclassical production theory. Using the concept of "production core," we determine stable task assignments and the corresponding factor-return distributions and then examine how these equilibrium outcomes respond to neutral technical progress pertaining to a particular worker or to all factors. Technical progress that is uniform in all factors will not alter equilibrium micro-matching. Technical progress of the labor-augmenting type may (i) cause a "turnover" by destroying existing stable task assignments and creating new stable task assignments, (ii) generate a richer pattern of wage redistribution than that under labor-improving technical progress in Neoclassical production theory, and (iii) create "spillover" effects from the innovating worker to his/her potential matching machines and his/her directly and indirectly competing workers. The possibility of turnovers and the extent to which factor returns

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are redistributed depend on the value of the current matches, the extent of outside threats from latent technologies, and the size of technical progress.

Keywords Micro-matching · Stable assignment · Technical progress · Turnover

JEL Classification C71 · C78 · D20 · D33

1 Introduction

Rooted in the pivotal work by von Neumann (1953), two-sided micro-matching theory has become an important organizing framework for studying production, marriage, and college admissions games. In contrast with the macroeconomic search and matching literature [see a comprehensive survey by Rogerson et al. (2005)], micro-matching models focus on patterns of task assignments within a microeconomic unit or market (say, a firm or a specific market, such as schools and marriage). The basic analysis is largely completed in the seminal pieces by Shapley and Shubik (1972); Owen (1975); Crawford and Knoer (1981) and Roth and Sotomayor (1990).[1] Generally speaking, the determination of micro-matching can be viewed as a linear assignment problem, where the concept of stable assignments is adopted to determine the equilibrium.

In this vein of micro-matching, our paper focuses on production games of two-sided micro-matching between heterogeneous workers and heterogeneous machines. Our paper generalizes the existing micro-matching literature by incorporating an important element in Neoclassical production theory—technical progress. We provide a complete comparative-static analysis on how the patterns of micro-matching and the resulting factor-return distributions respond to labor-improving technological advancements. In particular, we are able to characterize: (i) what size of technological progress leads to a "turnover" that destroys existing stable task assignments and creates new stable task assignments; (ii) how the equilibrium payoffs change when there is technical progress, with or without a turnover; and, (iii) what are the additional insights one can get on these issues from this two-sided micro-matching model as opposed to the Neoclassical model. Notably, although we have restricted our attention to production theory, the two-sided matching structure constructed in the present paper can be easily applied to the college admission game and the marriage game (e.g., see the classic by Gale and Shapley 1962, as well as more recent studies by Damiano and Li 2007 and Pais 2008). This can be done by regarding technical progress considered here as improvements in the quality of students/schools (college admission) or men/women (marriage).

More specifically, we construct a model with labor-improving technical progress where the production technology is described by two-sided micro-matching between a finite number of heterogeneous workers and a finite number of heterogeneous machines. The value of output can vary for any particular pair of worker and machine. Using the concept of "production core" *a la* von Neumann (1953), we determine "stable task assignments" that describe the pattern of micro-matching between

[1] Strictly speaking, this class of models features n-by-m disjoint agents with continuous payoffs.

workers and machines associated with manifest technologies. Those not in the production core represent latent technologies, which become "outside alternatives" to stable task assignments. Thus, any relative advancements in such technologies can potentially change the nature of micro-matching between workers and machines. The consideration of outside alternatives in constructing an equilibrium is a crucial feature in game-theoretic models including those two-sided matching models mentioned above. However, this feature is omitted in Neoclassical production theory that only accounts for manifest technologies. In our paper, we illustrate that, in determining the equilibrium after a particular type of technical progress, both manifest and latent technologies play important roles. Upon determining the equilibrium, we proceed with a complete characterization of the distribution of factor returns. We then undertake a thorough examination of how labor-improving technical progress that is pertaining to a particular worker regardless of the matching machine may influence stable task assignments within each production unit and the resulting redistribution of factor returns.

Our main findings can be summarized below. Theorem 1 shows that technical progress of the labor-augmenting type may cause a turnover by destroying existing stable task assignments and creating new stable task assignments. For such a turnover to occur, the size of this technical progress must be larger than a lower bound which is decreasing in the incremental change in the value of the match that the innovating worker forms and increasing in the incremental changes in the values of all the matches that all the other workers form. Theorem 2 characterizes how the equilibrium payoffs change in response to such a labor-improving technical progress, with or without a turnover. The properties obtained in our micro-matching framework contain the Neoclassical features, including a factor-return redistribution similar to one under Harrod-neutral technical progress in Neoclassical theory. Theorem 3 illustrates what are the additional insights one can get from the two-sided micro-matching model as opposed to the Neoclassical model. In particular, with a Neoclassical Harrod-neutral distribution in which the innovating worker acquires the entire productivity gain, such a gain may be greater or smaller than the direct incremental value of production created by the manifest technology associated with the innovating worker, contrasting with Neoclassical theory. Moreover, technical progress of the labor-augmenting type for a particular worker can create "spillover effects" on factor returns to the innovating worker's potential mates (machines) and his/her directly and indirectly competing workers. While this type of technical progress causes disadvantages for the worker losing his/her machine to the innovating worker relative to the worker taking over an innovating worker's old mate and others indirectly competing workers, it grants the innovating worker's new mate advantages over the innovating worker's old mate and other potential mates.

The remainder of the paper is organized as follows. Section 2 constructs a two-sided micro-matching framework with heterogeneous workers and machines, and defines stable assignments and stable factor-return distributions. Section 3 defines the equilibrium based on the concept of production core and the sets of equilibrium distributions associated with two types of neutral technical progress. In Sect. 4, we study how each type of neutral technical progress may influence stable assignments and equilibrium factor-return distributions. Finally, we summarize the main properties established and propose some avenues of future research in the concluding section.

2 The basic framework

We focus on characterizing two-sided micro-matching between workers and machines within each production unit, say, a firm. There are $n \geq 2$ workers and $m \geq 2$ machines. Denote the set of workers within the firm of our consideration as L, the set of machines as K, and the set of "agents" as $A = L \cup K$. A *task* (i, j) consists of a pair of worker and machine (ℓ_i, k_j) where $\ell_i \in L$ and $k_j \in K$. Each task creates a payoff $v_{ij} \geq 0$ and the payoff matrix $\mathcal{V} = \left(v_{ij}\right)$ summarizes all the payoffs associated with different tasks.

An *assignment*, denoted by μ, is a list of tasks with no worker or machine involving in more than one task:

$$\mu = \{(i, j) | \text{each } i \text{ and } j \text{ is matched at most once, for } i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m\} \tag{1}$$

Thus, an assignment describes *potential* micro-matching between workers and machines. Denote the *set of all possible assignments* as $[\mu]$. The *value* of production associated with an assignment μ is measured by,

$$V(\mu) = \sum_{(i,j)\in\mu} v_{ij} \tag{2}$$

Obviously, our production technology satisfies the Neoclassical constant-returns-to-scale property, that is, increasing the numbers of workers and machines proportionately will lead to an increase in the value of production in the same scale.

Definition 1 An *efficient* assignment $\mu^e \in [\mu]$ is an assignment such that $V(\mu^e) \geq V(\mu)$ for all $\mu \in [\mu]$.

Let w_i and z_j denote the returns to worker $\ell_i \in L$ and to machine $k_j \in K$, respectively.

Definition 2 A *distribution of factor returns* $X(\mu) = (w_1, \ldots, w_n, z_1, \ldots, z_m)$ is one such that $w_i \geq 0, z_j \geq 0$, and $w_i + z_j = v_{ij}$ for all $(i, j) \in \mu$. The *set of factor-return distributions* is denoted as $[X]$.

Definition 3 A *stable* assignment is an efficient assignment $\mu^* \in [\mu]$ associated with factor-return distributions $X^*(\mu^*) = (w_1, \ldots, w_n, z_1, \ldots, z_m)$ such that

$$w_i + z_j \geq v_{ij} \quad \text{for all } (i, j) \in \mu \text{ and for all } \mu \in [\mu] \tag{3}$$

$$w_i + z_j = v_{ij} \quad \text{for } (i, j) \in \mu^*. \tag{4}$$

where $X^*(\mu^*)$ is called *stable factor-return distributions*. The *set of stable assignments* is denoted as $[\mu^*]$ and the *set of stable factor-return distributions* is denoted as $[X^*]$.

The set of stable assignments describes the pattern of micro-matching between workers and machines with *manifest* technologies. Other assignments represent *latent* technologies, which are associated with outside alternatives to currently stable assignments. Specifically, for all $(i, j) \in \mu$ and for all $\mu \in [\mu]$, the assignment (i, j) is stable if $w_i + z_j \geq v_{ij}$. When $w_i + z_j > v_{ij}$, the task (i, j) is associated with a latent technology; when $w_i + z_j = v_{ij}$, it is associated with a manifest technology.

3 Production core, technical progress and distribution

We define the concept of equilibrium using *production core*, represented by the set of stable factor-return distributions $[X^*]$ that correspond to the set of stable assignments $[\mu^*]$. Since $[\mu^*]$ summarizes all manifest production activities, $V(\mu^*)$ measures the *GNP* associated with $\mathcal{V}$ from the production side:

$$V(\mu^*) = \sum_{(i,j)\in\mu^*} v_{ij} \tag{5}$$

By measuring GNP from the factor income side based on stable factor distributions, we have:

$$V(\mu^*) = \sum_{i\in L} w_i + \sum_{j\in K} z_j \tag{6}$$

The considerations of technical progress do not change the fact that production core contains the solution of the underlying linear assignment problem. Applying the von Neumann–Birkhoff duality theorem, one can focus on the dual concerning the distribution of factor returns and then prove the non-emptiness of production core, in terms of both equilibrium factor-return distributions in the dual problem and stable task assignments in the primal problem.

Lemma 1 $[\mu^*] \neq \emptyset$ *and* $[X^*] \neq \emptyset$.

Proof See Dantzig (1963); Shapley and Shubik (1972), and a sharper proof provided by Roth and Sotomayor (1990, Sects. 8.1 and 8.2).

We next define the concept of technical progress. Throughout this paper, we use the notation "tilde" to denote the post-technical progress entity. We are particularly interested in a form of technical progress that is labor-improving.

Definition 4 Consider technical progress of size $\lambda > 1$. It is called *labor-i uniform* if $\tilde{v}_{ij} = \lambda v_{ij}$ for all j and $\tilde{v}_{i'j} = v_{i'j}$ for all $i' \neq i$ and for all j.

Labor-i uniform technical progress can be regarded as a form of labor-augmenting technical progress where the origin of technical progress is due exclusively to worker i.

The technical progress is called *significant* if the post-technical progress stable assignment differs from the pre-technical progress stable assignment. We can now formalize the notion of turnover.

Definition 5 Let μ^* be the initial stable assignment and $\tilde{\mu}^*$ be a stable assignment after significant technical progress. Then, a turnover from μ^* to $\tilde{\mu}^*$ is given by

$$\tau\left(\mu^*, \tilde{\mu}^*\right) = \left\{\left((i, j), (\tilde{i}, \tilde{j})\right) \in \left(\mu^*, \tilde{\mu}^*\right) \mid \text{either } i = \tilde{i}, \textit{ or } j = \tilde{j}, \text{ but not both}\right\}$$

It is noted that $\tau(\mu^*, \tilde{\mu}^*) \neq \emptyset$ whenever $\tilde{\mu}^* \neq \mu^*$. It is clear that, when a turnover occurs under labor-i uniform technique progress, i must be the common player with $((i, j), (i, j')) \in \tau(\mu^*, \tilde{\mu}^*)$, where j and j' are i's old and new mates, respectively.

Because we are interested in the speed of turnovers, it is necessary to specify the dynamic process of technical progress. We consider that technical progress arrives at a Poisson rate $\eta > 0$ with a scaling factor $Z > 1$. That is, technology improves by a factor Z over an average length of period $1/\eta$. Then, letting $g = \frac{\dot{\lambda}(t)}{\lambda(t)}$ denote the rate of technical progress, we have:

$$\lambda(t) = \lambda(0)e^{gt}, \text{ with } \quad g = \eta \ln(Z) > 0 \tag{7}$$

Despite the use of simple dynamics to govern technical progress, the resulting changes in stable matches is rather complex. In particular, it may lead to turnover and cause an *endogenously determined* set of players to alter their matches (see Sect. 4).

Remark 1 It is clear that one may define *machine-j uniform* technical progress in the sense that $\tilde{v}_{ij} = \lambda v_{ij}$ for all i and $\tilde{v}_{ij'} = v_{ij'}$ for all $j' \neq j$ and for all i. Because machine-j uniform technical progress is mathematically isomorphic to labor-i uniform technical progress if one interchanges indexes i and j, we will not discuss the machine-j uniform case further but simply note that our results concerning labor-i uniform technical progress will immediately apply to this case. Moreover, one may also define *overall uniform* technical progress in the sense that $\tilde{v}_{ij} = \lambda v_{ij}$ for all i and j. It is straightforward that a turnover would never arise under overall uniform technical progress, which will not be analyzed for the sake of brevity.[2]

In the remainder of the paper, we shall assume, without loss of generality, that $n = m$. This is because, if, say, $n > m$, we can always add $n - m$ dummy machines that yield zero payoffs into $\mathcal{V}$. It is evident that the state of the equilibrium is not changed by this act.

Before turning to the subsequent section where we will establish useful properties associated with overall uniform technical progress and labor-i uniform technical progress, we would like to provide a 2-by-2 example to illustrate the working of our two-sided matching framework.

Example 1 Consider a 2-by-2 case with $v_{11} = 5$, $v_{12} = 3$, $v_{21} = 7$, and $v_{22} = 6$. Under labor-2 uniform technical progress of size $\lambda > 1$, the stable assignment becomes $\mu^* = \{(1, 1), (2, 2)\}$. The vertices of the set of stable distributions of factor returns can be derived in Table 1 below.[3]

[2] After overall uniform technical progress, each side of (3) and (4) will be multiplied by λ, which can be all cancelled out. Thus, the set of stable assignments remains unchanged.

[3] The solution technique used herein follows that in Dantzig (1963) and Shapley and Shubik (1972). An alternative is to adopt the deferred acceptance algorithm developed by Crawford and Knoer (1981).

Table 1 Stable factor-return distributions

w_1	w_2	z_1	z_2	vertex
3	6	2	0	A
4	6	1	0	B
0	2	5	4	C
0	3	5	3	D

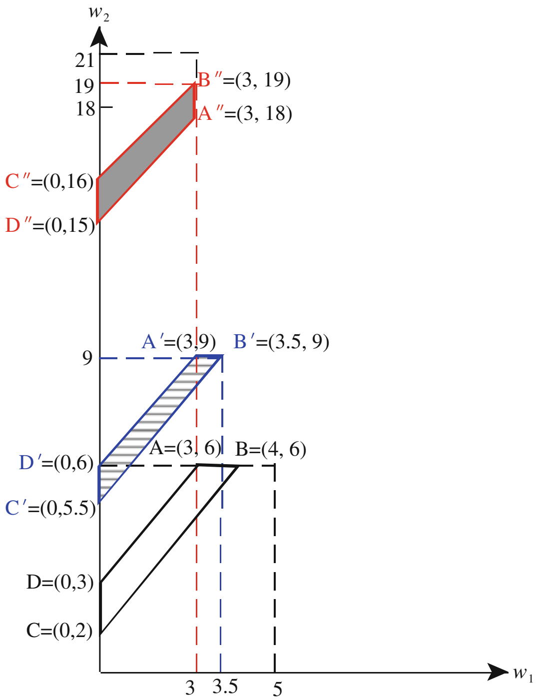

Fig. 1 A 2-by-2 example—Labor-2 uniform technical progress and turnover

The two-dimensional projection of the set of stable factor-return distributions in (w_1, w_2) space is plotted in Fig. 1. When there is weak labor-2 uniform technical progress with $1 < \lambda < 8/7$, the incremental returns to the innovating worker l_2 are low compared to his/her created value of production. When labor-2 uniform technical progress is moderately strong with $8/7 < \lambda < 2$, there are no turnovers but the incremental returns to the innovating worker l_2 now exceeds his/her created value of production as a result of his/her increasing threat of breaking the current match. When labor-2 uniform technical progress is sufficiently large with $\lambda > 2$, the previously stable assignment $\mu^* = \{(1, 1), (2, 2)\}$ is destroyed and a new stable assignment $\widetilde{\mu}^* = \{(1, 2), (2, 1)\}$ is created.

4 Equilibrium analysis

In this section, we will characterize how technical progress may influence micro-matching and distribution of factor returns. It is particularly interesting when labor-*i*

Table 2 Circle and non-circle players after a turnover

	ℓ_1	ℓ_2	$\cdots$	ℓ_{i-1}	ℓ_i	ℓ_{i+1}	$\cdots$	ℓ_s	ℓ_{s+1}	$\cdots$	ℓ_n
k_1	$(1, 1) \in [\mu^*]$							$(s, 1) \in [\widetilde{\mu}^*]$			
k_2	$(1, 2) \in [\widetilde{\mu}^*]$	$\ddots$									
$\vdots$		$\ddots$	$\ddots$								
k_{i-1}			$\ddots$	$(i-1, i-1) \in [\mu^*]$							
k_i				$(i-1, i) \in [\widetilde{\mu}^*]$	$(i, i) \in [\mu^*]$						
k_{i+1}					$(i, i+1) \in [\widetilde{\mu}^*]$	$(i+1, i+1) \in [\mu^*]$					
$\vdots$						$\ddots$	$\ddots$				
k_s							$\ddots$	$(s, s) \in [\mu^*]$			
k_{s+1}									$(s+1, s+1) \in [\mu^*] \cap [\widetilde{\mu}^*]$		
$\vdots$										$\ddots$	
k_n											$(n, n) \in [\mu^*] \cap [\widetilde{\mu}^*]$

uniform technical progress results in turnover, destroying existing stable assignments and creating new stable assignments.

Denote $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{2n})$ and the distribution before and after technical progress to all factors but the innovator (labor-i) as $[X^*_{-i}]$ and $[\tilde{X}^*_{-i}]$. Changed matches as a result of turnover form a *circle* (of modulus n), denoted by C. Call those players involved in these changing matches as *circle players*, with the set of circle players denoted by A_c; thus, the set $A \backslash A_c = L \cup K \backslash A_c$ contains all non-circle players (obviously, all dummy agents added to the original economy with zero payoffs must be non-circle players). Thus, for non-circle players, there is no turnover of their task assignments. This decomposition enables us to focus on establishing properties concerning mainly circle players.

To simplify the illustration, we reorder all agents in such a way that ℓ_i and $k_{j(i)}$ are matched prior to labor-i overall uniform technical progress of size $\lambda > 1$ and ℓ_i and $k_{j(i)+1}$ are matched after turnover (a circle of modulus n). Without loss of generality, we relabel $j(i) = i$ and delineate the micro-matching for circle and non-circle players in Table 2. Here, each agent is labeled by an element in the index set $I = \{1, 2, \ldots, n\}$. The entry of the diagonal, $(1, 1), \ldots, (i, i), \ldots, (s, s)$, are elements of $[\mu^*]$, whereas the entry, $(1, 2), \ldots, (i, i+1), \ldots, (s, 1)$, are elements of $[\widetilde{\mu}^*]$. Thus, each circle player can now be indexed by an element in $I_c = \{1, 2, \ldots, s\} \subset I$. For the non-circle players, $\ell_{s+1}, \ldots, \ell_n, k_{s+1}, \ldots, k_n$, their equilibrium matches remain unchanged. Call the configuration that summarizes the sequence of rematches of circle players as the *rematching circle*.

The following example helps illustrating the idea under the rematching circle.

Example 2 Consider a 3-by-3 case that augments the 2-by-2 case given in Sect. 3 with workers (a, b, c) and machines (a', b', c'), where $v_{aa'} = 5$, $v_{ba'} = 7$, $v_{ca'} = 3$, $v_{ab'} = 3$, $v_{bb'} = 6$, $v_{cb'} = 0.5$, $v_{ac'} = 1$, $v_{bc'} = 8$, and $v_{cc'} = 4$. The pre-turnover matching features a stable assignment $\mu^* = \{(a, a'), (b, b'), (c, c')\}$ with $V(\mu^*) = 15$. Under labor-b uniform technical progress of size $\lambda = 2.5$, turnover occurs where the new stable assignment becomes $\widetilde{\mu}^* = \{(a, b'), (b, c'), \left(c, a'\right)\}$ with $V(\widetilde{\mu}^*) = 26$. Now, think of the sequence of rematches to begin with the innovative worker b using language in graph theory: (i) b "severs" the match with b' to "link" with c'; (ii) c' severs the match with c who then links with a'; and, finally, (iii) a' severs the match with a who in turn links with b'. Thus, by denoting sever and link relationships by $\rightsquigarrow$ and $\rightarrow$, respectively, the configuration, $b \rightarrow c' \rightsquigarrow c \rightarrow a' \rightsquigarrow a \rightarrow b'$, constitutes the rematching circle. The relabelling process mentioned above involves naming b as ℓ_1, b 's new match c' as k_1, c''s severed match c as ℓ_2, c's new match a' as k_2, a''s severed match a as ℓ_3, and a's new match b' as k_3.

Define the minimal size of labor-i uniform technical progress for a turnover associated with a circle C to occur as:

$$\Lambda_i(C) = \frac{\sum_{a \neq i, a \in I_c} (v_{a,a} - v_{a,a+1})}{v_{i,i+1} - v_{i,i}} \tag{8}$$

We are now ready to determine explicitly when a turnover would occur:

Theorem 1 (Greatest Lower Bound and Critical Time of Turnover) *When a turnover associated with a circle C occurs under labor-i uniform technical progress of size $\lambda > 1$, it must exceed the greatest lower bound:*

$$\lambda \geq \Lambda_i(C). \tag{9}$$

Under the technical progress specified in (7) with $\lambda(0) = 1$, the critical calendar time for turnover to occur is given by,

$$T_i(C) = \frac{\ln \Lambda_i(C)}{\eta \ln (Z)} \tag{10}$$

Proof Applying the GNP Eq. (5) both before and after the turnover, we have:

$$V(\mu^*) = \sum_{a \in I} v_{aa} = \sum_{a \in I_c} v_{aa} + \sum_{a \in I \setminus I_c} v_{aa} \tag{11}$$

$$V(\tilde{\mu}^*) = \sum_{a \in I} \tilde{v}_{a,a+1} = \sum_{a \neq i, a \in I_c} v_{a,a+1} + \sum_{a \in I \setminus I_c} v_{a,a+1} + \lambda v_{i,i+1} \tag{12}$$

Equating $V(\tilde{\mu}^*)$ with $V(\mu^*)$ yields the critical value $\Lambda_i(C)$ as specified in (8). Substituting the equality in (8) into (7) yields (10). □

One may regard $1/T_i(C)$ as a measure of the "speed of turnover"—a larger value means a turnover can occur even with small sized labor-i uniform technical progress.

Theorem 1 indicates that the speed of turnover depends crucially on the Poisson arrival rate of the new technology (η) and scaling factor of technical progress (Z), the relative productivity of the relevant matches ($\frac{v_{a,a}-v_{a,a+1}}{v_{i,i+1}-v_{i,i}}$), as well as the endogenously determined set of circle players whose matches are affected by the technological change (I_c).

Remark 2 Condition (9) in Theorem 1 is necessary and sufficient for a turnover to occur. Thus, $\Lambda_i(C)$ gives the greatest lower bound of technical progress for turnover. It is worthwhile to note that although both stable assignments and equilibrium payoffs need not be unique, each stable assignment and the associated equilibrium payoff are compatible with each other and together form a complete specification of the equilibrium state of the economy. This is because all such stable assignments must give the *same* equilibrium value of production or GNP, both before and after the turnover (i.e., identical $V(\mu^*)$ and $V(\tilde{\mu}^*)$). The critical value given in (8) is the greatest lower bound for the associated initial set of stable assignments and the post-technical progress set of stable assignments. Although we need to specify a particular turnover circle C in order to compute the greatest lower bound, any other possible circle would give us exactly the same critical value, because this critical value is obtained purely from the GNP equations (see the proof of Theorem 1). Thus, the result is more general than one may think on first reading, as it holds *independently* of the particular initial assignment under consideration.

Remark 3 While it is straightforward that both the arrival rate and scaling factor of technical progress (η and Z) raise the speed of turnover, the effect of the relative productivity of the matches on the speed of turnover deserves further illustration.

- (i) The term $v_{i,i+1} - v_{i,i}$ is the incremental value accrued for the innovating worker ℓ_i to give up the existing match and to subsequently create a circle of new matches; it can thus be viewed as a measure of the temptation for the innovating worker to alter the match.
- (ii) The term $\sum_{a\neq i, a\in I_c}(v_{a,a} - v_{a,a+1})$ summarizes the aggregate opportunity costs of separating the existing matches of other circle players, which measures the level of resistance to new matches.
- (iii) Consider two payoff matrixes $\mathcal{V} = (v_{hj})$ and $\mathcal{V}' = (v'_{hj})$, where $v'_{hj} = \theta v_{hj}$ with $\theta > 1$ for all h, j. Then, $\Lambda_i(C)|_{\mathcal{V}'} = \Lambda_i(C)|_{\mathcal{V}}$, that is, the minimal size of labor-i uniform technical progress for a turnover associated with a circle C to occur is unchanged when the payoffs are uniformly higher by the same proportion.
- (iv) Consider two payoff matrixes $\mathcal{V} = (v_{hj})$ and $\mathcal{V}' = (v'_{hj})$, where $v'_{ij} = \theta v_{ij}$ with $\theta > 1$ and $v'_{hj} = v_{hj}$ for all j and $h \neq i$. Then, $\Lambda_i(C)|_{\mathcal{V}'} < \Lambda_i(C)|_{\mathcal{V}}$, that is, the minimal size of labor-i uniform technical progress for a turnover associated with a circle C to occur is lower under $\mathcal{V}'$ in which worker ℓ_i is uniformly more productive.

When the temptation for the innovating worker to alter the match is relatively high compared to the level of resistance to rematches, turnover can occur with a relatively small minimal size $\Lambda_i(C)$ and relatively shorter critical calendar time.

Upon examining the critical point of turnover, we next turn to studying the properties of equilibrium factor-return distributions under our general micro-matching setup (Theorem 2) and the Neoclassical counterpart of Harrod-neutral distributions (Theorem 3). For notational convenience, we shall relabel all circle players in such a way that the pre-turnover matches are indexed by (a, a) and the post-turnover matches by $(a, a+1)$.

Theorem 2 (Equilibrium Distribution After Labor-i Uniform Technical Progress) *Consider* $a \in I_c, (a, a) \in [\mu^*]$ *and* $(a, a+1) \in [\tilde{\mu}^*]$. *Define* $\Delta w_i = \tilde{w}_i - w_i$ *and* $\Delta v_{ij} = \tilde{v}_{ij} - v_{ij}$. *Let* $(w, z) \in [X^*], (\widetilde{w}, \widetilde{z}) \in [\widetilde{X}^*]$ *and* $|A_c| = s$. *Then,*

(i) *the equilibrium distribution to workers satisfies:*

$$\begin{aligned} \Delta w_s + \Delta v_{11} \leq \Delta w_1 \leq \Delta w_2 + \Delta v_{12} \\ \Delta w_{s-1} + \Delta v_{ss} \leq \Delta w_s \leq \Delta w_1 + \Delta v_{s1} \\ \Delta w_{i-1} + \Delta v_{ii} \leq \Delta w_i \leq \Delta w_{i+1} + \Delta v_{i,i+1} \quad \text{if } i \notin \{1, s\} \end{aligned} \tag{13}$$

and,

$$\Delta w_s \leq \Delta w_1 \text{ if } i \notin \{1, s\} \quad \text{and} \quad \Delta w_{a-1} \leq \Delta w_a \quad \text{for all } a \in I_c \backslash \{i, i+1\} \tag{14}$$

(ii) *the equilibrium distribution to machines satisfies:*

$$\begin{aligned} \Delta z_1 - \Delta z_s \leq \Delta v_{s1} - \Delta v_{ss} \\ \Delta z_{i+1} - \Delta z_i \leq \Delta v_{i,i+1} - \Delta v_{ii} \quad \text{if } i \neq s \end{aligned} \tag{15}$$

and,

$$\Delta z_1 \leq \Delta z_s \text{ if } i \neq s \quad \text{and} \quad \Delta z_{a+1} \leq \Delta z_a \text{ for all } a \in I_c \backslash \{i\} \tag{16}$$

Proof To prove part (i), we use (3) and (4) to write,

$$w_i + z_{j(i)} = v_{i,j(i)}, \quad w_i + z_{j(i)+1} \geq v_{i,j(i)} \tag{17}$$

$$\tilde{w}_i + \tilde{z}_{j(i)+1} = \tilde{v}_{i,j(i)+1}, \quad \tilde{w}_i + \tilde{z}_{j(i)} \geq \tilde{v}_{ij} \tag{18}$$

By eliminating $z_{j(i)}$ and $\tilde{z}_{j(i)+1}$, using $z_j = v_{ij} - w_i$ and $\tilde{z}_j = \tilde{v}_{i-1,j} - \tilde{w}_{i-1}$ and relabelling $j(i) = i$, (17) and (18) imply:

$$w_i - w_{i-1} \leq v_{ii} - v_{i-1,i} \tag{19}$$

$$w_{i+1} - w_i \leq v_{i+1,i+1} - v_{i,i+1} \tag{20}$$

$$\tilde{w}_i - \tilde{w}_{i-1} \geq \tilde{v}_{ii} - v_{i-1,i} = \lambda v_{ii} - v_{i-1,i} = v_{ii} - v_{i-1,i} + \Delta v_{ii} \tag{21}$$

$$\begin{aligned} \tilde{w}_{i+1} - \tilde{w}_i \geq \tilde{v}_{i+1,i+1} - \tilde{v}_{i,i+1} = v_{i+1,i+1} - \lambda v_{i,i+1} = v_{i+1,i+1} - v_{i,i+1} \\ -\Delta v_{i,i+1} \end{aligned} \tag{22}$$

Utilizing (19) and (21), one obtains the first inequality in (13); combining (20) and (22) further yields the second inequality in (13). We can then obtain (14) by applying (13) to any worker $\ell_a \in A_c$ and by recognizing that $\Delta v_{aa} = \Delta v_{a,a+1} = 0$ for all $a \in I_c \backslash \{i\}$ and that circle players is a circle of modulus s.

Similarly, we can prove part (ii) by using (3) and (4) to obtain:

$$w_i + z_{j(i)} = v_{i,j(i)}, \quad w_{i-1} + z_{j(i)} \geq v_{i-1,j(i)} \tag{23}$$

$$\tilde{w}_{i-1} + \tilde{z}_{j(i)} = \tilde{v}_{i-1,j(i)}, \quad \tilde{w}_i + \tilde{z}_{j(i)} \geq \tilde{v}_{ij(i)} \tag{24}$$

By eliminating w_i and $\tilde{w}_i$ and and relabeling $j(i) = i$, (23) and (24) imply, respectively,

$$z_i - z_{i-1} \geq v_{i-1,i} - v_{i-1,i-1} \tag{25}$$

$$z_{i+1} - z_i \geq v_{i,i+1} - v_{ii} \tag{26}$$

$$\tilde{z}_{i-1} - \tilde{z}_i \geq \lambda \left(v_{i-1,i-1} - v_{i-1,i}\right) \tag{27}$$

$$\tilde{z}_i - \tilde{z}_{i+1} \geq \lambda \left(v_{ii} - v_{i,i+1}\right) \tag{28}$$

We can now combine (25) and (27) to get the inequality in (15) and combine (26) and (28) to yield an inequality in (16): $\Delta z_i \leq \Delta z_{i-1}$. We finally prove the remaining inequalities in (16) by applying (15) to $a \in I_c$ and by recognizing that $\Delta v_{aa} = \Delta v_{a,a+1} = 0$ for all $a \in I_c \backslash \{i\}$ and that circle players is a circle of modulus s. □

Theorem 2 shows not only how labor-augmenting technical progress enhances the innovating worker's return, but also how it influences (i) the innovating worker's old and new mates (machines k_i and k_{i+1}, respectively), (ii) the innovating worker's direct competitors (worker ℓ_{i-1}, who takes over ℓ_i's pre-turnover matching machine k_i, and worker ℓ_{i+1}, who yields his/her pre-turnover matching machine k_{i+1} to ℓ_i after the turnover), and (iii) the innovating worker's indirect competitors (all other workers in the circle ℓ_a, $a \in I_c \diagdown \{i-1, i, i+1\}$). Of particular interest, Theorem 2 establishes the *spillover effects* of technical progress pertaining to worker ℓ_i and his/her old and new mates, which are further elaborated in the following.

Corollary 1 (Changes in the Returns to Non-innovating Workers) *Consider a game and a labor-i uniform technical progress given in Theorem 2. Then, the equilibrium distribution to workers satisfies:*

(i) *after a turnover, the incremental return to the innovating worker ℓ_i exceeds that to worker ℓ_{i-1}, who takes over ℓ_i's pre-turnover matching machine, by at least the incremental value of production accrued to i's pre-turnover stable assignment (i, i);*
(ii) *after a turnover, the incremental return to the innovating worker ℓ_i exceeds that to worker ℓ_{i+1}, who yields his/her matching machine to ℓ_i after the turnover, by no more than the incremental value of production accrued to ℓ_i's post-turnover stable assignment $(i, i+1)$;*
(iii) *the incremental returns to workers as a result of turnover increases along the rematching circle.*

Proof The first inequality of (13) implies $\Delta w_i - \Delta w_{i-1} \geq \Delta v_{ii}$, which yields property (i). The second inequality of (13) says $\Delta w_i - \Delta w_{i+1} \leq \Delta v_{i,i+1}$, thus yielding (ii). Property (iii) immediately follows from the inequalities given in (14). □

Corollary 2 (Changes in the Returns to Machines) *Consider a game and a labor- i uniform technical progress given in Theorem 2. Then, the equilibrium distribution to machines satisfies:*

(i) *after a turnover, the incremental return to the innovating worker's new mate k_{i+1} exceeds that to the innovating worker's old mate k_i by no more than the differential in the value of production between the new stable assignment $(i, i+1)$ and the old one (i, i);*
(ii) *the incremental returns to machines as a result of turnover decreases along the rematching circle.*

Proof The value of production differential in (15), $\Delta v_{i,i+1} - \Delta v_{ii}$, is always positive (otherwise, turnover would have not occurred). property (ii) immediately follows from (16). Such a redistribution must, however, be limited by the extent of technical progress. As a consequence, the differential between the incremental returns to machine k_{i+1} and that to machine k_i is bounded by $\Delta v_{i,i+1} - \Delta v_{ii}$, which proves (i). □

Intuitively, the worker yielding his/her pre-turnover mate to ℓ_i after the turnover (worker ℓ_{i+1}) is the "head" of the circle whereas the worker taking over ℓ_i's pre-turnover mate (worker ℓ_i) is the "tail" of the circle. Because the former suffers the most direct loss from turnover (directly crowded out by the innovating worker), his/her incremental return must be less than those less directly influenced at a later position of the circle C. This gives the entire ordering of incremental returns along the rematching circle. As to machines, the innovating worker ℓ_i's new mate (machine k_{i+1}) is the head of the circle whereas ℓ_i's old mate (machine k_i) is the tail of the circle. Being the innovating worker's new mate would receive the greatest benefits, thus giving the entire ordering of incremental returns along the rematching circle.

Remark 4 The patterns of increasing incremental returns to workers and decreasing incremental returns to machines along the circle (14) and (16), are parallel to those in the entry game considered by Mo (1988) along the turnover chain. This is not unanticipated because, in either game, there is a set of circle/chain players formed in response to technical progress (a turnover circle) or entry (a turnover chain). However, in our model, we have additional properties concerning the redistribution of factor returns as stated in (13) and (15) (as well as in Theorem 3 below).

Example 3 Consider the 2-by-2 case given in Example 1 with $v_{11} = 5$, $v_{12} = 3$, $v_{21} = 7$, and $v_{22} = 6$. The critical value of turnover is $\Lambda_2(C) = 2$. If such labor-2 uniform technical progress arrives twice a year ($\eta = 2$) with an expansion rate of $5\%(Z = 1.05)$, then worker ℓ_2 improves at an annual rate of $2(\ln 1.05) \approx 9.76\%$ and turnover will occur after $\frac{\ln 2}{2(\ln 1.05)} \approx 7.1$ years. Consider such a technical progress of size $\lambda = 2.5 > \Lambda_2(C)$. It is interesting that not only the innovative worker ℓ_2 receives incremental returns (e.g., $\min \widetilde{w}_2 = 12.5 > 2.5 \min w_2 = 5$), but the innovative worker's new match k_1 also gains extra returns (e.g., $\max \widetilde{z}_1 = 17 > \max z_2 = 8$). That is, there is a positive spillover effect to the innovative worker's new match.

Finally, we would like to contrast our results with findings in Neoclassical production theory. To facilitate such comparison, we select a *neoclassical Harrod-neutral distribution* (cf. Allen 1938, Harrod 1939, and Wan 1971) associated with labor-i uniform technical progress in the sense that $\tilde{w}_a = w_a$ for each $a \neq i$, $a \in I_c$, $\tilde{w}_i > w_i$, and $\tilde{z}_a = z_a$ for each $a \in I_c$. By the proof of Theorem 2, it is not difficult to find a stable distribution of factor returns such that $\tilde{w}_i > w_i$. Thus, our results contain those in Neoclassical production theory. In this special case, we can further pin down explicitly the incremental return to the innovating worker.

Theorem 3 (Equilibrium Neoclassical Harrod-neutral Distribution) *Consider* $a \in I_c$, $(a, a) \in [\mu^*]$ *and* $(a, a+1) \in [\tilde{\mu}^*]$. *Define* $\Delta w_i = \tilde{w}_i - w_i$, $\Delta v_{ij} = \tilde{v}_{ij} - v_{ij}$, *and* $\Delta V = V(\tilde{\mu}^*) - V(\mu^*)$. *Let* $(w, z) \in [X^*]$. *A Neoclassical Harrod-neutral distribution satisfies:*

(i) *(marginal productivity)*

$$\Delta w_i = \Delta V \tag{29}$$

(ii) *(incremental return to the innovating worker)*

$$\begin{aligned}\Delta w_i = (\lambda - 1)v_{i,i+1} + \left[\left(v_{i-1,i} + v_{i,i+1}\right) - \left(v_{ii} + v_{i+1,i+1}\right)\right] \\ - \sum_{a \in A_c, a \neq i, i-1} \left(v_{a+1,a+1} - v_{a,a+1}\right)\end{aligned} \tag{30}$$

Proof To construct a Neoclassical Harrod-neutral distribution, we set $\tilde{w}_a = w_a \forall a \neq i$ and $\tilde{z}_a = z_a$ and let $j = i$ (by relabeling). Applying (6) both before and after the turnover, we have:

$$V(\mu^*) = \sum_{a \in L} w_a + \sum_{a \in K} z_a \quad \text{and} \quad V(\tilde{\mu}^*) = \tilde{w}_i + \sum_{a \in L, a \neq i} w_a + \sum_{a \in K} z_a \tag{31}$$

which can be combined to yield (29) in part (i).

To prove part (ii), we utilize (17), (21) and (22) to derive

$$w_{a+1} - w_a = v_{a+1,a+1} - v_{a,a+1} \quad \forall a \neq i \text{ or } i-1, \ a \in I_c \tag{32}$$

Next, (17), (21) and (22) together yield:

$$(\Delta v_{ii} - \Delta w_i) + v_{ii} - v_{i-1,i} \leq w_i - w_{i-1} \leq v_{ii} - v_{i-1,i} \tag{33}$$

$$(\Delta v_{i,i+1} - \Delta w_i) + v_{i+1,i+1} - v_{i,i+1} \leq w_{i+1} - w_i \leq v_{i+1,i+1} - v_{i,i+1} \tag{34}$$

Applying (5) and (6) both before and after the turnover, we get:

$$V(\mu^*) = \sum_{a\in A_c} v_{aa} + \sum_{a\notin A_c} v_{aa} = \sum_{a\in L} w_a + \sum_{a\in K} z_a \tag{35}$$

$$V(\tilde{\mu}^*) = \sum_{a\in A_c, a\neq i} v_{a,a+1} + \sum_{a\notin A_c} v_{a,a+1} + \lambda v_{i,i+1} = \tilde{w}_i + \sum_{a\in L, a\neq i} w_a + \sum_{a\in K} z_a \tag{36}$$

These can then be combined with (32)–(34) to obtain (30). □

Theorem 3 delivers two sharp results. The first is on marginal productivity, illustrating the absorption of the total productivity gain by the innovating worker. The second further solves explicitly incremental returns to the innovating worker, which need not be equal to the incremental value of production from the innovating worker's post-turnover new match.

Remark 5 The results established in Theorem 3 deserve further comments.

(i) (Marginal Productivity) One may regard part (i) of Theorem 3 as a special form of marginal productivity theory (*à la* Wicksteed 1894 and Wicksell 1900) in the context of the Neoclassical Harrod-neutrality distribution that resembles the "no-surplus" condition in general equilibrium theory (cf. Ostroy 1980.[4] Its meaning is straightforward. As a result of worker ℓ_i's innovation, the aggregate surplus accrued is ΔV. When the total productivity gain is completely internalized by the innovating worker ($\Delta w_i = \Delta V$), there must be no surplus accrued to the remaining agents (*i.e.*, $\tilde{w}_a = w_a$ for each $a \neq i, a \in I_c$ and $\tilde{z}_a = z_a$ for each $a \in I_c$). Conversely, when the no-surplus condition holds, the total productivity gain must be fully absorbed by the innovating worker.

(ii) (Incremental Returns to the Innovating Worker) Part (ii) of Theorem 3 suggests that labor-i uniform technical progress benefits the innovating worker ℓ_i only when

  (a) such labor-augmenting technical progress is sizable (i.e., λ is large),
  (b) the value of production associated with the new match is sufficiently higher than that with the old match (i.e., $(v_{i-1,i} + v_{i,i+1}) - (v_{i,i} - v_{i+1,i+1})$ is large),
  (c) the change in the value of production from rematches for other circle players is sufficiently low [i.e., $\sum_{a\in A_c, a\neq i,i-1} \left(v_{a+1,a+1} - v_{a,a+1}\right)$ is small].

  Purely from the viewpoint of the post-turnover manifest technology associated with the innovating worker, the incremental return to the innovating worker may be greater than or less than the direct incremental value of production created by the manifest technology associated with the post-turnover stable task assignment $(i, i+1)$. That is, $\Delta w_i - \Delta v_{i,i+1} = \Delta w_i - (\lambda - 1)v_{i,i+1}$ may be positive or negative, in contrast with its counterpart in Neoclassical production theory where incremental return to the innovating worker must be equal to the direct

[4] For a comprehensive overview of marginal productivity theory, see Mo (2010).

incremental value of production created by the manifest technology associated with the innovating worker. This different finding results from two special features of our two-sided micro-matching framework. One is the explicit account for the role of latent technologies as outside alternatives. Another is the explicit account for the spillover from the innovating worker to his/her potential mates and his/her directly and indirectly competing workers. Thus, even under the Neoclassical Harrod-neutral distribution scheme, our results are much richer than those obtained in Neoclassical production theory.

5 Concluding remarks

We have constructed a two-sided micro-matching framework with heterogeneous workers and machines, allowing for on-going technical progress. Some punch-line properties have been established.

Labor-i uniform technical progress may result in turnover, destroying existing stable assignments and creating new ones. Turnover arises if the size of technical progress is sufficiently small, or if the resistance from the existing matches is strong, relative to the innovator's productivity gain.

After a turnover as a result of labor-i uniform technical progress, the incremental return to the innovating worker ℓ_i (i) exceeds that to the worker who takes over ℓ_i's pre-turnover matching machine by at least the incremental value of production accrued to ℓ_i's pre-turnover match and (ii) exceeds that to the worker who yields his/her matching machine to ℓ_i after the turnover by no more than the incremental value of production accrued to ℓ_i's post-turnover match. Under the Neoclassical Harrod-neutral distribution scheme, the innovating worker acquires the entire productivity gain, which may be greater or less than the direct incremental value of production created by the manifest technology associated with the innovating worker's new match. In general, labor-i uniform technical progress creates spillovers in factor-return distributions to all other agents as a result of turnover. On the one hand, the incremental returns to other workers increase along the circle, ordered from the worker who yields his/her matching machine to ℓ_i to the worker who takes over ℓ_i's pre-turnover matching machine. On the other hand, the incremental returns to machines decrease along the circle, ordered from ℓ_i's new mate to ℓ_i's old mate.

Along these lines, one may study the equilibrium consequences of a non-neutral technical progress, which may involve only a single task or be localized to a subset of agents. Consider, for example, the case where labor-augmenting technical progress of different size is associated with more than one worker. One may express each new matching sequence as a circle, where some circles may connect with each other (i.e., different circles share with the same tail) and some may not (i.e., each circle has its independent tail). It is not difficult to see that the properties established in Theorems 2 and 3 can then be applied to each circle of new matches.

A second avenue that may be of interest is to measure formally labor skills/machine productivity and to consider the changes of micro-matching with ongoing technological advances. This can then be contrasted with the technological cycle obtained by Jovanovic (2009). In particular, one may examine whether matching is positive assortative in the sense that more skilled workers are matched with more productive machines. One may also examine the implications of skill-biased technical progress for wage inequality.

A third avenue is to examine another main issue of dynamics, namely, factor accumulation. In the two-sided micro-matching framework, accumulation of a particular factor can be viewed as entry of an identical twin of a particular agent (either a machine or a worker). The basic methodology established in this paper is, with the assistance from the setup in the entry game by Mo (1988), readily applied to this extension. Versions of magnification properties such as Jones-Rybczynski theorem may then be established in response to the expansion of a particular factor. Our results in Sect. 4.1 concerning wage inequalities can also apply here. For example, consider an entry of low skilled workers from less-developed economies to an advanced country as a consequence of globalization. Specifically, one can show that losses in marginal productivity will be greater than losses in marginal returns on low skilled workers (in absolute value). Thus, some low skilled workers may become unmatched and some skilled workers may have less desirable matches that can cause their wages to fall. As a result, the overall wage inequality need not be unambiguously widened. A third avenue is to generalize the one-to-one matching structure to many-to-one or many-to-many (cf. Roth 1984 and Crawford 1991), or to consider a general balanced matching game. The generalization to many-to-one matching may be useful for studying the behavior of firm with many workers, and the generalization to many-to-many matching may be particularly relevant to understanding the interactions between producing firms and outsourcing subcontractors. While such generalization has its merits, the reader should be warned that it can only be done at the expenses of losing many sharp comparative static results (particularly those stated in Theorems 2 and 3 above).

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