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Friedman meets Becker and Mulligan in a monetary neoclassical growth model

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Abstract We study a neoclassical growth model with the time preference determined by resources spent on imagining future pleasures along the line of Becker and Mulligan (Q J Econ 112:729–758, 1997). We introduce money into the economy via a cash-in-advance constraint and study the effect of higher seignorage taxes or higher monetary growth rates on capital, consumption and welfare in the long run. We find that if the fraction of investment constrained by cash is smaller than a threshold, the negative-monetary-growth Friedman (The Optimum Quantity of Money and Other Essays, 1969) rule does not hold and the optimal inflation rate is positive. Calibrating our model yields a mild optimal inflation rate per annum with a switch from zero inflation to optimal inflation creating a sizable welfare gain in terms of consumption equivalence.

Keywords Friedman rule · Endogenous time preferences · Inflation tax · Neoclassical growth model

JEL Classification E22 · E31

"*Patience is bitter, but its fruit is sweet.*" *Jean Jacques Rousseau* (1712–1778)

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1 Introduction

Patience has long been recognized as an important virtue in human history.[1] In economics, time preferences have played a fundamental role in theories of savings, investment and economic growth among many other issues. Yet, since Ramsey (1928), time preference rates are almost always taken as exogenous with little discussion concerning what determines their level. Although there have been efforts favoring endogenous time preferences, these time preferences are either a by-product of consumption choices (Uzawa 1968) or a by-product of fertility choices in the form of an intergenerational discount rate (Becker and Barro 1988).

Different from these above exogenous and endogenous time preferences, Becker and Mulligan (1997) proposed a time preference that is not a by-product of other choices. They postulated that consumers exert themselves and carry out activities in order to influence the discount on future utilities. According to Becker and Mulligan (1997), the rate of a time preference is determined by the resources spent on imagining future pleasures, termed as "future oriented capital". Thus, Becker and Mulligan (1997) determined the time preference rate by relating it to resources spent on imagining future pleasures and put forward that the larger the resources are spent the more patient an individual is.

In their concluding remarks, Becker and Mulligan (1997, p. 754) pointed out several directions for future work in order to envisage the implications of endogenous time preferences as the result of resources spent on imagining future pleasures. Since then, many researchers have studied the consequences and implications of this type of time preference on the issues regarding addictions and health concerns (Bretteville-Jensen 1999), cultural transmissions and social status (Bisin and Verdier 2001), the formation of markets and institutions (Palacios-Huerta and Santos 2004), religious interactions (Bisin et al. 2004), and occupational choices and the spirit of capitalism (Doepke and Zilibotti 2008).

Although time preferences are important determinants of savings and capital accumulation, with the exceptions of Stern (2006) and Gong (2006), to the best of our knowledge, few researchers have studied the resulting effects on savings and capital accumulation. Stern (2006) studied a one-sector neoclassical growth model with resources spent on imagining future pleasures. He characterized the uniqueness and multiplicities of a steady state, and investigated the stability property in a series of examples with parametric functions for utilities, time preferences and productions. Gong (2006) examined a one-sector neoclassical growth model with a money-in-the-utility function and resources spent on imagining future pleasures. Gong (2006) found a negative long-run relationship between money and capital as in Stockman (1981) and Wang and Yip (1992) wherein money affected their economies through a cash-in-advance constraint on consumption and investment. This line of literature thus

[1] For example, St. Augustine (354–430) noted "patience is the companion of wisdom;" Ralph Waldo Emerson (1803–1882) counseled "adopt the pace of nature; her secret is patience;" and Albert Einstein (1879–1955) quipped "it's not that I'm so smart; it's just that I stay with problems longer." Similar maxims can be found throughout Asia to whit a Chinese proverb cautions " if you are patient in a moment of anger, you will escape a hundred days of sorrow" along with its Indian proverb companion "for the friendship of two, the patience of one is required."

indicates a negative long-run relationship between money and capital under either the case when money enters the utility and agents spend resources to influence their time preferences or the case when money constrains consumption and investment.

In this paper, we investigate the robustness of the relationship between money and capital in the long run when cash constrains consumption and investment and agents spend resources on imagining future pleasures. Our framework thus combines the model with resources spent on imagining future pleasures by Stern (2006) and the model with a cash-in-advance (henceforth, CIA) constraint by Stockman (1981) and Wang and Yip (1992).[2] Specifically, we study a standard one-sector growth model wherein consumers spend resources on imagining future pleasures with a CIA constraint binding consumption and investment. At an instantaneous point in time, the representative agent produces and allocates goods to three activities: (i) consumption that increases current utility; (ii) savings that accumulates capital and increases future utility; and, (iii) resources spent on imagining future pleasures that increase one's own appreciation of the future, which lowers the discount and raises discounted future utility.

In this model, we find that a higher seignorage tax increases capital and consumption if the fraction of investment constrained by cash is smaller than a threshold. Intuitively, in the special case when the CIA constraint binds consumption only and then the fraction of investment constrained by cash is zero, monetary growth raises the price of consumption relative to both investment and spending on imagining future pleasures. Thus, monetary growth leads agents to substitute away from consumption toward spending on imagining future pleasures which decreases the time preference rate and in turn increases savings and capital in the long run. As a general case when a fraction of investment is also bound by cash, monetary growth has an additional direct negative effect on investment. To the extent when the fraction of investment constrained by cash is smaller than a threshold, the direct negative effect is smaller and thus monetary growth increases capital in the long run. As a result, the Friedman (1969) rule of a negative monetary growth rate does not hold and the optimal inflation rate is positive.[3] By calibrating our model the optimal inflation rate per annum is 2.09% and a switch of monetary policies from zero inflation to optimal inflation creates a sizable welfare gain that is 6.96% in terms of consumption equivalence per annum.

Mixed empirical evidence about the relationship between money and capital has relevance for our work. Employing 5-year average data across countries, Bruno and Easterly (1998) found a negative relationship between inflation and growth for high inflation countries. Using annual post-war data for 32 countries, however, Karras (1993) showed that monetary growth had a probably neutral effect on output in the long run, but the effect is positive in the short run. Indeed, in a large sample of postwar economies, Bullard Keating (1995) found a long-run positive relationship between

[2] Other growth models adopted a CIA constraint, including Englund and Svensson (1988), Dotsey and Sarte (2000), Wang and Wen (2007) and Chen and Guo (2008), Crucini et al. (2010) and Lu et al. (2011a,b).

[3] According to Friedman (1969) doctrine, an optimal monetary policy would involve a steady state contraction of the money supply at a rate that brings the nominal interest rate down to zero. In a growth model, this means a monetary growth rate equal the negative time preference rate.

inflation and output in low inflation countries. Utilizing annual time-series data for G-7 countries, Ericsson et al. (2001) found that inflation and output were co-integrated and typically output and inflation were positively related in these co-integrating relationships for most countries. The result in our model suggests that resources spent on imagining future pleasures and the fraction of investment constrained by cash may be one of the reasons underlying the ambiguous relationship between money/inflation and capital.

Chang and Tsai (2003) also found a positive relationship between money and capital when the fraction of investment constrained by cash is smaller than a threshold. Their positive relationship comes from the social statue effect in utility emerged as a by-product of capital and real balances.[4] The positive relationship in our model emerges because agents choose to spend resource on imagining future pleasures that lowers the time preference rate.

The paper proceeds as follows. Section 2 sets up a model, investigates the equilibrium conditions, and studies the existence of the uniqueness of steady state. Section 3 analyzes long-run effects of permanent monetary growth and analyzes the optimum quantity of money. In Sect. 4, we offer concluding remarks. Finally, Sect. 5 is the Appendix.

2 The model

Our model integrates the models by Wang and Yip (1992) and Stern (2006). Time is continuous. The lifetime utility of the representative agent is

$$U = \int_0^{\infty} u(c_t) X_t \, dt, \tag{1}$$

where c is consumption and u is an instantaneous utility function. The discount factor is $X_t \equiv \exp[-\int_0^t \rho_\tau d\tau]$, where ρ is the instantaneous discount rate.

The agent may engage in some activity or abide via sacrifice in order to imagine future pleasures and increase the appreciation of the future. Resources spent on imagining future pleasures decrease the discount rate. Then, the discount function is $\rho = \rho(s)$ and the discount factor may be rewritten as

$$\dot{X}_t = -\rho(s_t) X_t, \quad \text{with } X_0 \text{ given}, \tag{2}$$

where s is the resource cost spent on imagining future pleasures.

The utility function $u(c)$ and the discount function $\rho(s)$ have the following properties.

[4] Chang and Tsai (2003) was a Wang and Yip (1992) model with a utility increasing in the social status effect which is represented by wealth and thus the sum of capital and real balances. In their model, when the fraction of investment constrained by cash is smaller than a threshold in their model, a higher inflation rate makes agents substitute away from money to capital in order to maintain social status.

Assumption 1 (i) $u(c) > 0$ *and* $u'(c) > 0 > u''(c)$ *for any* $c > 0$;

(ii) $\rho(s) > 0$ *and* $\rho'(s) < 0 < \rho''(s)$ *with* $\rho''(s)/\rho'(s) - \rho'(s)/\rho(s) \leq 0$ *for any* $s > 0$;

(iii) $u''(c)u(c)[\rho'(s)^2 - \rho''(s)] - [u'(c)\rho'(s)]^2 > 0$ *for any* $c > 0$ *and* $s > 0$ *and* $u'(0)/u(0) > -\rho'(s)$ *for all* s.

Assumption 1 (i) postulates a strictly concave utility function with a positive and decreasing marginal utility of consumption. The requirement $u(c) > 0$ is necessary in order to have a positive marginal benefit of spending on imagining future pleasures so the spending on imagining future pleasures is positive in a steady state. Note that under $u(c) > 0$, it follows that $u''(c)/u'(c) - u'(c)/u(c) < 0$. This property is consistent with the empirical observation that the intertemporal elasticity of substitution for consumption is usually less than 1 (e.g., Ogaki and Reinhart 1998).[5]

Assumption 1 (ii) requires a positive discount rate. Moreover, it requires a negative and decreasing marginal discount of spending on imagining future pleasures. The assumption $\rho''(s)/\rho'(s) - \rho'(s)/\rho(s) \leq 0$ requires that the discount be sufficiently convex.[6] These assumptions follow from Stern (2006) which was taken from Becker and Mulligan (1997).

Assumption 1 (iii) is technical conditions. The former part assures the discounted instantaneous utility to be jointly and strictly concave in c and s so that the Hamiltonian is strictly concave. Specifically, let the discounted instantaneous utility be $W(c, s) = u(c)X$. Then, it is straightforward to obtain $W_{cc} = u''(c)X < 0$, $W_{ss} = u(c)X[(\rho'(s))^2 - \rho''(s)] < 0$ and $W_{cc}W_{ss} - (W_{sc})^2 = X^2\{u''(c)u(c)[\rho'(s)^2 - \rho''(s)] - [u'(c)\rho'(s)]^2\} > 0$. The latter part restricts the shape of the Keynes-Ramsey condition so that the steady state is unique.

The production function is $y = f(k)$, where y is output per capita and k is capital per capita, with $k(0)$ given initially, and, for simplicity, is assumed not to depreciate. We assume a standard concave production function with the Inada condition as follows.

Assumption 2 $f'(k) > 0 > f''(k)$, $f(0) = 0$, $\lim_{k\to 0} f'(k) = \infty$, *and* $\lim_{k\to\infty} f'(k) = 0$.

The representative agent faces the following budget constraint.

$$\dot{m}_t = f(k_t) - \pi_t m_t + v_t - c_t - s_t - I_t, \tag{3a}$$

where I is gross investment, m is real money balances, π is the inflation rate, and v is real lump-sum transfers. The budget constraint indicates that available resources may be consumed, spent on imagining future pleasures, or saved. Savings are held in the form of either investment or real balances. Nominal money is initially given and grows at a constant rate μ. We assume that transfers are made by government which are financed by monetary growth; thus, in aggregates $v_t = \mu m_t$.

[5] For example, the CES utility $(c^{1-\sigma} - 1)/(1 - \sigma)$.

[6] For example, a discount function used in Stern (2006) is $\rho(s) = \gamma(s)^{-\theta} - 1$ which meets the condition $\rho''(s)/\rho'(s) - \rho'(s)/\rho(s) \leq 0$ when θ is smaller and γ is larger.

Capital is accumulated as follows.

$$\dot{k}_t = I_t, \quad \text{with } k_0 \text{ given.} \tag{3b}$$

The representative agent also faces the following general CIA constraint.

$$c_t + \varphi I_t \leq m_t, \quad 0 \leq \varphi \leq 1. \tag{3c}$$

Based on the literature on endogenous time preferences, the form of spending on imagining future pleasures would lower time discounting at a diminishing rate, which can be viewed as a "reduced form" to satisfy all the regularity conditions in the recursive preference literature summarized by Becker and Boyd (1997). Spending on imagining future pleasures yields higher pleasures for future consumption through enhancing "savings in kind" which is not subject to the CIA constraint.[7] Thus, in (3c) we do not consider a cash constraint on spending on imagining future pleasures.

In (3c), if $\varphi = 0$, only consumption is bound by cash. This is the constraint used in Lucas (1980). If $\varphi = 1$, both consumption and investment are equally constrained by cash. This is the constraint used in Stockman (1981). Finally, if $0 < \varphi < 1$, this is the constraint used in Wang and Yip (1992).

Thus, our model includes (2)–(3c). Our model is reduced to the Wang and Yip (1992) model if there is no spending on imaging future pleasures (namely, $s_t = 0$) and thereafter the time preference rate ρ is exogenously given and further to Stockman (1981) if $\varphi = 1$ and to Lucas (1980) if $\varphi = 0$. Alternatively, our model is reduced to the Stern (2006) model if there is no money ($m_t = 0$) and thereafter the CIA constraint in (3c) is not binding.[8]

Let us remark on the utility representation (1) and (2). According to Stern (2006), the representation has two main ways of interpretation, depending on whether one views the optimal program as a dynastic family or as a single individual with an infinite lifetime.

In a dynastic family with each generation living for two periods, childhood and adulthood, the discount factor is the degree to which generation t cares for generation $t + 1$. The linkage between the parent and the child is endogenously modeled. The resources spent in the appreciation of the future s_t then stand for actions that the parent takes in order to reinforce the connection with the child. The degree to which the parent engages and commits himself to the nurturing and rearing of the child would certainly be a determinant in the intensity of the relationship. Spending more time to read and to play with the child could also be a determinant in the strength of the relationship. As a result, spending in the parent–child relationship would cost the parent either foregone production or current resources.

[7] "Savings-in-kind" activities that increase the imagination of future pleasures include, among others, spending more time to read and to play with the child and activities such as schooling/learning, religion, mortality/health, and time spent in trying to appreciate the future. These activities are not subject to the CIA constraint.

[8] Stern (2006) is in a discrete time.

An alternative explanation is an individual with an infinite lifetime. The agent maximizes the sum of current utility and discounted sum of utilities of the remainder of his life. The discount factor is applied to the individual's own utility in the future, as opposed to that of a dynastic family model that is applied to descendants. Resources spent on imagining future pleasures are used to increase the individual's own appreciation of the future. As put forth by Becker and Mulligan (1997, pp 2 and 10), activities such as schooling/learning, mortality/health, religion and time spent in trying to appreciate the future all increase the imagination of future pleasures. Like Becker and Mulligan (1997), our specification is in line with the second method of interpretation. Different from Becker and Mulligan (1997) wherein the model is in a discrete time and the agent has a finite horizon, our model is in a continuous time and the agent has an infinite horizon.

2.1 Optimization conditions

The representative agent's problem is to maximize (1) subject to (2) and (3a)–(3c), taking as given the monetary growth rate, transfers, initial capital and initial nominal money holdings. The discounted utility is strictly concave. The constraints (2) and (3a) are concave and the constraints (3b) and (3c) are linear. Thus, the Hamiltonian function is concave. The necessary conditions are

$$u'(c_t) = \lambda_{mt} + \xi_t, \tag{4a}$$

$$-\theta_t \rho'(s_t) = \lambda_{mt}, \tag{4b}$$

$$\lambda_{kt} = \lambda_{mt} + \varphi \xi_t, \tag{4c}$$

$$\dot{\lambda}_{kt} = -f'(k_t)\lambda_{mt} + \rho(s_t)\lambda_{kt}, \tag{4d}$$

$$\dot{\lambda}_{mt} = [\pi_t + \rho(s_t)]\lambda_{mt} - \xi_t, \tag{4e}$$

$$\dot{\theta}_t = -u(c_t) + \theta_t \rho(s_t), \tag{4f}$$

along with the transversality constraints:

$$\lim_{t\to\infty} \lambda_{kt} X_t k_t = 0, \quad \lim_{t\to\infty} \lambda_{mt} X_t m_t = 0 \quad \text{and} \quad \lim_{t\to\infty} \theta_t X_t = 0,$$

where $\lambda_{kt} > 0, \lambda_{mt} > 0$ and $\theta_t > 0$ denote the (current-valued) co-state variables associated with capital, real balances and the discount factor, respectively, and $\xi_t > 0$ denotes the (current-valued) multiplier of the CIA constraint.

Optimal conditions (4a)–(4c) are for consumption, spending on imagining future pleasures and investment. For example, (4a) equates the marginal utility of consumption to the marginal cost of consumption in order to determine optimal consumption. The conditions in (4d)–(4f) are Euler equations that govern how the shadow prices of capital, real balances and the discount factor change over time, respectively.

2.2 Equilibrium

An equilibrium is a time path $\{c_t, s_t, k_t, m_t, \lambda_{kt}, \lambda_{mt}, \xi_t, \theta_t, \pi_t\}$ that satisfies optimal conditions (4a)–(4f), the binding CIA constraint (3c),[9] the money market clearance condition,

$$\dot{m}_t = (\mu - \pi_t)m_t, \tag{5a}$$

and the goods market clearance condition, which, using (3a), (3b) and (5a), is

$$\dot{k}_t = f(k_t) - c_t - s_t. \tag{5b}$$

Below, we explain how the equilibrium is determined. First, if we substitute ξ_t in (4a) into (4c), we obtain

$$\lambda_{kt} = \varphi u'(c_t) + (1-\varphi)\lambda_{mt} \equiv \lambda_{kt}(c_t, \lambda_{mt}). \tag{6a}$$

Next, differentiating (4a) with respect to time, with the use of (4c), (4d) and (6a), leads to the following modified Keynes-Ramsey rule,

$$\dot{c}_t = \frac{1}{u''(c_t)}\left\{\left(1-\frac{1}{\varphi}\right)\dot{\lambda}_{mt} + \frac{1}{\varphi}\rho(s_t)\lambda_{kt}(c_t, \lambda_{mt}) - \frac{1}{\varphi}f'(k_t)\lambda_{mt}\right\}. \tag{6b}$$

Moreover, differentiating (4b) with respect to time, with the use of (4f), yields

$$\dot{s}_t = \frac{-1}{\theta_t \rho''(s_t)}\left\{\dot{\lambda}_{mt} - \rho(s_t)\lambda_{mt} - u(c_t)\rho'(s_t)\right\}, \tag{6c}$$

which is a variant of the Keynes-Ramsey rule for spending on imagining future pleasures.

Furthermore, as (3b) and (5b) indicate $f(k_t) = c_t + I_t + s_t$, the binding CIA constraint suggests $m_t = (1-\varphi)c_t + \varphi f(k_t) - \varphi s_t$. If we differentiate this relationship with respect to time and use (5a), the inflation rate is equal to the difference between the money growth rate and an expression that is a function of $\dot{c}_t$, $\dot{k}_t$ and $\dot{s}_t$. Substituting in $\dot{k}_t$ from (5b), $\dot{c}_t$ from (6b), and $\dot{s}_t$ from (6c), along with $\dot{\lambda}_{mt}$ from (4e), the inflation rate is determined by

$$\pi_t = \mu - \frac{(1-\varphi)\dot{c}_t + \varphi f'(k_t)\dot{k}_t - \varphi\dot{s}_t}{(1-\varphi)c_t + \varphi f(k_1) - \varphi s_t} = \pi(c_t, k_t, s_t, \lambda_{mt}). \tag{6d}$$

[9] Following Lucas (1980) and Stockman (1981), we assume the CIA constraint is binding in equilibrium. Roughly speaking, this requires that the monetary growth rate be greater than or equal to the discounted marginal rate of substitution between consumption in two consecutive points in time.

Finally, substituting in ξ_t from (4c) to (4e), together with (6a) and (6d), yields

$$\dot{\lambda}_{mt} = \lambda_{mt}\left[\rho(s_t) + \frac{1}{\varphi} + \pi(c_t, k_t, s_t, \lambda_{mt})\right] - \frac{1}{\varphi}\lambda_{kt}(c_t, \lambda_{mt}). \tag{6e}$$

Thus, the equilibrium system is simplified to five equations, including (5a), (5b), (6b), (6c) and (6e) and solves for five equilibrium paths: $c_t, k_t, s_t, \lambda_{mt}$, and m_t. The equilibrium system is block-recursive: when (5b), (6b), (6c) and (6e) simultaneously determine the paths of c_t, k_t, s_t and λ_{mt}, the path of m_t is determined by (5a). The paths of the remaining variables, $\lambda_{kt}, \xi_t, \pi_t$ and θ_t, are in turn determined by (6a), (4c), (6d) and (4b).

In a steady state, $\dot{c}_t = \dot{k}_t = \dot{s}_t = \dot{\lambda}_{mt} = \dot{m}_t = 0$. To determine steady state, first, $\pi = \mu$, according to (5a), and thus the inflation rate is equal to the monetary growth rate. Then, using (6a) and (6e), we rewrite (5b), (6b) and (6c), respectively, as follows.

$$f(k) = c + s, \tag{7a}$$

$$f'(k) = \rho(s)\left\{1 + \varphi[\rho(s) + \mu]\right\}, \tag{7b}$$

$$u'(c)\rho(s) = -\rho'(s)u(c)[1 + \rho(s) + \mu]. \tag{7c}$$

The steady-state conditions thus include the otherwise standard commodity clearance condition (7a) and the otherwise standard Keynes-Ramsey rule (7b) except for spending for imaging future pleasures. Equation (7c) is new to our otherwise standard growth model. While the left-hand side of (7c) is the steady-state marginal cost of spending on imagining future pleasures, the right-hand side of (7c) is the steady-state marginal benefit of spending on imagining future pleasures in steady state. It is worth noting that $u(c) > 0$ in Assumption 1(i) assures a positive marginal benefit of spending on imagining future pleasures such that it is equal to a positive marginal cost of spending on imagining future pleasures.

Denote $\Psi \equiv -\frac{\rho(s)}{u(c)}[u''(c)u(c) - (u'(c))^2] > 0$ and $\Omega \equiv \frac{u(c)(1+\rho(s)+\mu)[\rho''(s)\rho(s)-(\rho'(s))^2]}{\rho(s)} + (\rho'(s))^2 u(c) > 0$.[10] By the implicit function theorem, (7c) gives the following relationship

$$s = s(c, \mu), \tag{8a}$$

in which

$$s_c \equiv \frac{\partial x}{\partial c} = \frac{\Psi}{\Omega} > 0 \quad \text{and} \quad s_\mu \equiv \frac{\partial s}{\partial \mu} = -\frac{\rho'(s)u(c)}{\Omega} > 0. \tag{8b}$$

[10] In signing Ψ and Ω, we use Assumption 1(i) and (ii), respectively.

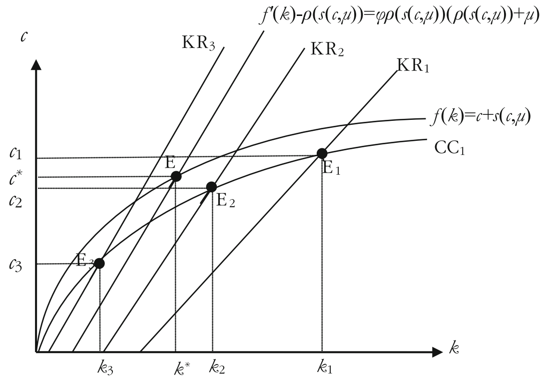

Fig. 1 Existence and uniqueness of steady state and comparative-static effects

Substituting (8a) into (7a) and (7b) yields, respectively,

$$f(k) = c + s(c, \mu), \tag{9a}$$

$$f'(k) = \rho(s(c, \mu)) + \varphi\rho(s(c, \mu))[\rho(s(c, \mu)) + \mu]. \tag{9b}$$

Thus, (9a) and (9b) simultaneously determine the values of k and c in steady state. When k and c are solved, we can determine s from (7c), m from (3c) and λ_m from (6e). For simplicity, (9a) is referred to as the CC locus and (9b) as the KR locus. We will analyze the steady state in the (k, c) plane.

In the (k, c) plane, the slope of the CC locus in (9a) is positive,

$$\left.\frac{dc}{dk}\right|_{\mathrm{CC}} = \frac{f'(k)}{1 + s_c} > 0. \tag{10a}$$

Intuitively, a higher capital increases output. Consumption and spending on imagining future pleasures need to increase in order to clear the commodity market. Under Assumption 2, the CC locus starts from the origin and is increasing and concave. See Fig. 1.

To analyze the slope of the KR locus, we start with the special case when investment is not constrained by cash as it was in Lucas (1980).

A Special Case: $\varphi = 0$.

In this special case, the KR locus in (9b) is $f'(k) = \rho(s(c, \mu))$. In the (k, c) plane, the slope of the KR locus is positive,

$$\left.\frac{dc}{dk}\right|_{\mathrm{KR}} = \frac{f''(k)}{\rho'(s)s_c} > 0. \tag{10b}$$

Intuitively, a higher capital decreases the marginal product of capital. In optimum, consumption must increase in order to increase spending on imagining future pleasures so as to decrease the discount rate. See Fig. 1. [11]

The positively slopping KR locus may be steeper or flatter than the CC locus. The following condition assures that the positively sloping KR locus is steeper than the CC locus.

Condition S: $-\rho'(s) < \left(1 + \frac{1}{s_c}\right) \frac{-f''(k)}{f'(k)}$.

Condition S requires that the convex discount function $\rho(s)$ be flatter than the production function. Condition S is necessary to obtain a negative determinant of the Jacobean matrix in order to assure the saddle path stability.

Recall $\rho(s) > 0$ in Assumption 1(ii). Thus, when $c = 0$, (9b) gives

$$f'(k_0) = \rho(s(0, \mu)) + \varphi\rho(s(0, \mu))[\rho(s(0, \mu)) + \mu] > 0,$$

which indicates $k_0 > 0$ for any $\varphi \geq 0$. Thus, when $c = 0$, the KR locus starts from a positive k. See Fig. 1. Thus, there exists a unique steady state (k^*, c^*). See E in Fig. 1, the intersection of the KR locus and the CC locus.

General Case: $\varphi > 0$.

Next, in the general case, a fraction of investment is also constrained by cash. The KR locus in (9b) includes an additional term associated with φ. The slope of the KR locus is still positive.

$$\left.\frac{dc}{dk}\right|_{\text{KR}} = \frac{f''(k)}{\Phi s_c} > 0, \tag{10c}$$

where $\Phi \equiv \rho'(s)[1 + \varphi(2\rho(s) + \mu)] < 0$.

To make sure that the KR locus is steeper than the CC locus, it is required that

Condition SS: $-\rho'(s)[1 + \varphi(2\rho(s) + \mu)] < (1 + \frac{1}{s_c})\frac{-f''(k)}{f'(k)}$.

Like Condition S, Condition SS requires that the convex discount function $\rho(s)$ be flatter than the production function. Condition SS is necessary to obtain a negative determinant of the Jacobean matrix in order to assure the saddle path stability. Condition SS is reduced to Condition S if $\varphi = 0$. Under Condition SS, Condition S is automatically met.

The KR locus starts from a positive k because of the Inada condition. Thus, there exists a unique steady state. In Fig. 1, the intersection of the KR locus and the CC locus is the unique steady state. See the unique steady state (k^*, c^*) at E in Fig. 1.

We summarize the steady-state results as follows.

Proposition 1 *In a standard one-sector optimal growth model with spending on imagining future pleasures wherein consumption and a fraction of investment is constrained*

[11] In the case when the time preference is exogenous, the KR locus is vertical.

by cash, under Assumptions 1 and 2 and Condition SS, there exists a unique steady state.

It is worth noting the stability condition. Gong (2006) modeled money through a utility function and thus the stability condition can be analyzed in terms of a 3 × 3 system. In our model, money enters the economy through the cash constraint and there is the shadow price of the cash constraint. The most simplified dynamic system for analyzing the stability condition is in terms of (5b), (6b), (6c), and (6e) that involve c_t, k_t, s_t and λ_{mt}. This dynamic system includes one predetermined state variable. Thus, one negative eigenvalue is required to guarantee a saddle path toward the unique steady state.

In the Appendix 5.1, we take linear Taylor's expansion of the dynamic system near the steady state. We obtain the following determinant of the Jacobean matrix

$$\Xi^{\varphi} = \frac{-\lambda_m^* c^*}{[c^*\theta^*\rho''(s^*) + \lambda_m^*]u''(c^*)}\hat{\Delta}^{\varphi},$$

where

$$\begin{aligned}\hat{\Delta}^{\varphi} = f''(k^*)\Big\{&u(c^*)\rho''(s^*)(1+\rho(s^*)+\mu) - u''(c^*)\rho(s^*) + u(c^*)\rho'(s^*)^2 \\ &-u'(c^*)\rho'(s^*)[\rho(s^*)+\mu]\Big\} - f'(k^*)\rho'(s^*)[1+\varphi(2\rho(s^*)+\mu)] \\ &\times[u''(c^*)\rho(s^*) + u'(c^*)\rho'(s^*)] < 0.\end{aligned}$$

Under Condition SS, $\hat{\Delta}^{\varphi} < 0$ and the determinant is negative. This indicates that there are either one or three negative eigenvalues. In the case when $\varphi > 0$, it is difficult to pin down the number of negative eigenvalues to be exactly one or three. In our quantitative exercises in Sect. 4, we verify that there is only one eigenvalue and thus the equilibrium path toward the steady state is a saddle. However, we have analytically shown in the Appendix 5.1 that in the case of $\varphi = 0$, we simplify the characteristic polynomial of the Jacobean matrix and prove that the number of negative eigenvalues is exactly one. In this case, there is a unique saddle path toward the steady state.

3 Long-run effects of monetary growth and optimum quantity of money

This section analyzes the long-run effect of monetary growth and the optimum quantity of money.

3.1 Long-run effects of monetary growth

When the monetary growth rate μ is increased, the CC locus shifts rightwards,

$$\left.\frac{dk}{d\mu}\right|_{CC} = \frac{s_\mu}{f'(k)} > 0. \tag{11}$$

Intuitively, a higher monetary growth rate increases the price of consumption relative to spending on imagining future pleasures which increases the demand for spending on imagining future pleasures and thus, the demand for commodity. To maintain the commodity market clearance, capital needs to increase in order to increase the output supply, thereby shifting the CC locus rightwards.

A Special Case: $\varphi = 0$.

For the KR locus, in the special case when $\varphi = 0$, an increase in the monetary growth rate shifts the KR locus rightwards,

$$\left.\frac{dk}{d\mu}\right|_{\mathrm{KR}} = \frac{\rho'(s)s_{\mu}}{f''(k)} > 0. \tag{12a}$$

The reason is that a higher monetary growth rate increases spending on imagining future pleasures which decreases the rate of time preferences. This encourages individuals to save more, thereby shifting the KR locus rightwards.

As a result of the rightward shift of both the CC locus and the KR locus, capital is unambiguously increasing in the long run.

$$\frac{dk}{d\mu} = \frac{\rho'(s)s_{\mu}}{\Delta} > 0, \tag{12b}$$

where $\Delta \equiv -f'(k)\rho'(s)s_c + f''(k)(1 + s_c) < 0$ under Condition S.

Intuitively, as the CIA constraint binds consumption only, a higher monetary growth rate increases the price of consumption relative to spending on imagining future pleasures, making agents to substitute away from consumption toward spending on imagining future pleasures. As agents are more patient now, it is optimal to increase savings. As a result, capital increases in the long run.

The effect on consumption is ambiguous, depending on relative rightward shifts in these two loci. We obtain the following result.

$$\frac{dc}{d\mu} = \frac{s_{\mu}}{\Delta}\left[f'(k)\rho'(s) - f''(k)\right]\gtrless 0 \quad \text{if} \quad \frac{\rho'(s)}{f''(k)} \gtrless \frac{1}{f'(k)}. \tag{12c}$$

Under $\rho'(s)/f''(k) < 1/f'(k)$, the KR locus shifts rightwards less than the rightward shift of the CC locus and consumption decreases in the long run. See KR_2 and E_2 in Fig. 1. However, under $\rho'(s)/f''(k) > 1/f'(k)$, the KR locus shifts rightwards more than the rightward shift of the CC locus and consumption increases in the long run. See KR_1 and E_1 in Fig. 1. The condition $\rho'(s)/f''(k) > 1/f'(k)$ is equivalent to $-\rho'(s) > -f''(k)/f'(k)$, which requires the convex discount function to be steeper than the production function.

To summarize the above results,

Proposition 2 *In a standard one-sector optimal growth model with spending on imagining future pleasures when only consumption is constrained by cash, under Assumptions 1 and 2 and Condition S, higher monetary growth unambiguously increases capital and may increase consumption in the long run.*

Our above model is reduced to the model in Lucas (1980) if there is no spending on imaging future pleasures and thus $s = 0$. In such a Lucas (1980) model money does not affect capital and is thus neutral in the long run. In our model, because agents spend resources on imagining future pleasures and thus $s > 0$, higher monetary growth rate unambiguously increases capital in the long run. In particular, if the convex discount function is steeper than the production function, a higher monetary growth rate increases consumption in the long run.

General Case: $\varphi > 0$.

As a general case when a fraction of investment is constrained by cash, a higher monetary growth rate shifts the KR locus in the way as follows.

$$\left.\frac{dk}{d\mu}\right|_{\text{KR}} = \frac{\Phi s_\mu + \varphi\rho(s)}{f''(k)}. \tag{13a}$$

As the denominator is negative, the direction to which the KR locus shifts depends on the nominator in (13a). Obviously, the sign of the nominator in (13a) depends on the values of $\varphi > 0$ and $\Phi < 0$. Denote $\varphi_1 \equiv -\rho'(s)s_\mu/[\rho'(s)s_\mu(2\rho(s)+\mu)+\rho(s)]$. To assure $\varphi_1 > 0$, we assume that the convex discount function ρ is such that

Condition P: $-\rho'(s) < \frac{\rho(s)}{s_\mu[2\rho(s)+\mu]}$.

Under Condition P, $[\rho'(s)s_\mu(2\rho(s)+\mu)+\rho(s)] > 0$ and thus, $\varphi_1 > 0$.

First, in the case when $\varphi < \varphi_1$, then $\Phi s_\mu + \varphi\rho(s) < 0$, then, the KR locus shifts rightward to KR_1 or KR_2 in Fig. 1. As the CC locus shifts rightward to CC_1, capital unambiguously increases (E_1 or E_2) while consumption may increase (E_1) or decrease (E_2) in the long run.

Second, in the case when $\varphi > \varphi_1$, then $\Phi s_\mu + \varphi\rho(s) > 0$. Then, the KR locus unambiguously shifts leftward to KR_3 in Fig. 1. Given that the CC locus shifts rightward to CC_1, capital and consumption unambiguously decrease (E_3) in the long run.

The net effect on capital and consumption depends on the fraction of investment that is constrained by cash. Rewriting Condition SS as

$$\Delta^\varphi \equiv -f'(k)\Phi s_c + f''(k)(1+s_c) < 0.$$

Moreover, denote

$$\varphi_2 \equiv \frac{-\rho'(s)s_\mu}{[\rho'(s)s_\mu(2\rho(s)+\mu)+\rho(s)]+\rho(s)s_c} > 0 \quad \text{and}$$
$$\varphi_3 \equiv \frac{[-\rho'(s)f'(k)+f''(k)]s_\mu}{f'(k)[\rho'(s)s_\mu(2\rho(s)+\mu)+\rho(s)]}.$$

Under Condition P, $\varphi_2 > 0$. The effects on capital and consumption in the long run are

$$\frac{dk}{d\mu} = \frac{\Phi s_\mu + \varphi\rho(s)(1+s_c)}{\Delta^\varphi} \begin{matrix} > \\ < \end{matrix} 0, \quad \begin{matrix} \text{if } \varphi < \varphi_2 \\ \text{otherwise} \end{matrix} \tag{13b}$$

$$\frac{dc}{d\mu} = \frac{f'(k)(\Phi s_\mu + \varphi\rho(s)) - f''(k)s_\mu}{\Delta^\varphi} \begin{matrix} > \\ < \end{matrix} 0, \quad \begin{matrix} \text{if} \varphi < \varphi_3, \\ \text{otherwise.} \end{matrix} \tag{13c}$$

Thus, in the case when $\varphi > \varphi_1$, $\Phi s_\mu + \varphi\rho(s) > 0$, the nominators in (13b) and (13c) are unambiguously positive. Given $\Delta^\varphi < 0$, it follows that both capital and consumption are unambiguously decreasing in the long run.

Alternatively, when $\varphi < \varphi_1$, $\Phi s_\mu + \varphi\rho(s) < 0$ and the nominators in (13b) and (13c) may be positive or negative. Specifically, the nominator in (13b) is negative when $\varphi < \varphi_2$. As mentioned earlier, under $\varphi = 0$, (12c) indicates that under $-\rho'(s) > -f''(k)/f'(k)$, consumption increases in the long run. Under this condition, $\varphi_3 > 0$ and the nominator in (13c) is negative when $\varphi < \varphi_3$. Thus, consumption will increase in the long run when $\varphi > 0$ which is obtained if $-\rho'(s) > -f''(k)/f'(k)$. Apparently, $\varphi_2 < \varphi_1$ and $\varphi_3 < \varphi_1$. Thus, under $-\rho'(s) > -f''(k)/f'(k)$, if $\varphi < \varphi_0 \equiv min\{\varphi_{2,}\varphi_3\}$, then a higher monetary growth rate increases capital and consumption in the long run and thus the Friedman rule does not hold.

To summarize our results.

Proposition 3 *In an optimal growth model with spending on imagining future pleasures wherein consumption and a fraction of investment are constrained by cash, under Assumptions 1 and 2, Conditions SS and P and* $-\rho'(s) > -f''(k)/f'(k)$*, a higher monetary growth rate increases capital and consumption in the long run if the fraction investment constrained by cash is smaller than the threshold* $\varphi_0 \equiv min\{\varphi_2, \varphi_3\}$.

The intuition may be explained as follows. Our above model is reduced to the Wang and Yip (1992) model if there is no spending on imaging future pleasures and thus $s = 0$ and further to the Stockman (1981) model if $\varphi = 1$. In such models of Stockman (1981) and Wang and Yip (1992), there is only the direct negative effect of monetary growth emerged from the cash constraint on investment. Thus, monetary growth reduces capital in the long run. Now, agents spend resources on imaging future pleasures. As monetary growth induces agents to substitute away from consumption toward spending on imagining future pleasures and to become more patient, there is an indirect positive effect. When the fraction of investment constrained by cash is smaller than a threshold, the direct negative is small and is dominated by the indirect positive effect. As a result, monetary growth increases capital in the long run and the Friedman rule does not hold. In particular, if the convex discount function is steeper than the production function, a higher monetary growth rate increases consumption in the long run. Then, capital, output and consumption all increase in the long run.

Table 1 lists a summary of the effects of monetary growth that are obtained in the models of exogenous time preferences by Lucas (1980), Stockman (1981) and Wang and Yip (1992) and in our model of endogenous time preferences. In Table 1, we also demonstrate the results obtained in these conventional models with the Uzawa (1968) endogenous time preference that is affected by an agent's consumption. With the Uzawa endogenous time preference, in the Appendix 5.2 we obtain the conventional relationship between money and capital: a neutral relationship if only consumption is

Table 1 Summary of the results of monetary effects

	Only consumption enters CIA	Both consumption and investment enter CIA
A. Exogenous ρ	Lucas: money is superneutral	Stockman and Wang-Yip: money suppresses capital and output; Friedman rule holds
B. Endogenous ρ		
a. $\rho = \rho(c)$ (Uzawa)	Same as Lucas	Same as Stockman and Wang-Yip
b. $\rho = \rho(s)$ (Becker–Mulligan)	Money raises capital and may raise consumption; Friedman rule does not hold	Money may raise capital and consumption; Friedman rule does not hold

constrained by cash and a negative relationship if consumption and any small fraction of investment is constrained by cash.[12]

3.2 The optimum quantity of money

As the Friedman rule does not hold in our model, this subsection envisages the optimum quantity of money. The optimal growth rate of money is determined by maximizing the discounted lifetime utility of the representative agent in the long run,

$$U = \frac{u(c)}{\rho(s)}.$$

Differentiating the above long-run welfare with respect to μ and using (7c), (8a) and (8b) yields

$$\frac{dU}{d\mu} = \frac{-u(c)\rho'(s)}{\rho(s)^2}\left\{[1+\rho(s)+\mu+s_c]\frac{dc}{d\mu}+s_\mu\right\}. \tag{14}$$

While the term outside the large braces in (14) is positive, there are two terms inside the large braces in which the term (s_μ) is positive but the term associated with $dc/d\mu$ may be positive or negative. Thus, (14) may be larger than or equal to zero. In the case when $\varphi < \varphi_3$, our analysis in Sect. 3.1 indicates that (14) is unambiguously positive and thus a higher inflation tax always increases the welfare in the long run. Alternatively, if φ is larger than φ_3 but is not too large, then the term associated with $dc/d\mu$ is negative and can offset the term (s_μ) such that

$$[1+\rho(s)+\mu+s_c]\left(-\frac{dc}{d\mu}\right) = s_\mu. \tag{15}$$

[12] In the Appendix 5.2 we impose the standard assumption of $\rho'(c) > 0$ in the derivation of the stability condition. However, the relationship between money and capital mentioned above remain true if $\rho'(c) < 0$.

Then, there is an interior optimal inflation tax. In this situation, the optimal interior inflation tax μ^*, along with k^* and c^*, is characterized by (15) and (9a)–(9b). To summarize the result,

Proposition 4 *In an optimal growth model with spending on imagining future pleasures wherein consumption and a fraction of investment are constrained by cash, under Assumptions 1 and 2, Conditions SS and P and $-\rho'(s) > -f''(k)/f'(k)$, the Friedman rule may not hold and there exists a positive optimal growth rate of money.*

To offer a quantitative assessment about the optimum quantity of money, we calibrate our model in a steady state to reproduce some key features representative of the US economy at annual frequencies. We use the popular Cobb-Douglas production function: $f(k) = Ak^{\alpha}$, where $0 < \alpha < 1$ is the share of capital and $A > 0$ is the coefficient of productivity. We employ the popular utility function with a constant *intertemporal elasticity of substitutability* (IES): $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, where $\sigma > 0$ is the reciprocal of the IES. [13] We take the discount function from Stern (2006): $\rho(s) = \gamma(s)^{-\theta} - 1$, $\gamma > 0$ and $\theta > 0$.

In calibration, the parameter A is normalized to unity. We choose the capital share in output at $\alpha = 0.32$ as in real business cycles studies. Empirical estimates of the IES for consumption are in general smaller than one and we set σ equal to 2 such that the IES for consumption is 1/2. There is little evidence concerning the fraction of investment that is constrained by cash. As benchmark parameterization, we choose $\varphi = 0.4$ which means that a little bit more than one-third of investment is constrained by cash.[14] Since negative inflation (i.e., deflation) is not a usual situation, we choose $\mu = 0$ as our baseline parameterization. By setting the annual discount rate ρ at 0.06 and $\theta = 0.04$, (7b) can be used to obtain benchmark value of k and then (7a) gives the relationship $c(s)$. Finally, (7c) and the value of ρ, along with the relationship $c(s)$, simultaneously calibrate the benchmark value of s and γ. We obtain $\gamma = 1.0406$ and $s = 0.6297$. The baseline parameterization enables us to obtain steady-state values and the level of welfare in the long run: $k = 11.3232$, $c = 1.5444$, $s = 0.6297$, $y = 2.1741$ and $U = 5.8746$.[15]

We then change the growth rate of money to see how a steady state and the level of welfare change. As it turns out, when the growth rate of money is increasing from zero, the levels of capital, spending on imagining future pleasures and consumption increase in a steady state, with the level of consumption reaching the top at a smaller monetary growth rate than the former two variables. See Fig. 2.

An increase in consumption and an increase in spending on imagining future pleasures both increase the welfare and point to a positive optimal monetary growth rate. Our exercises indicate that the level of welfare peaks at $\mu^* = 0.0209$, after which the

[13] To assure $u > 0$, our utility form restricts c >1. Alternatively, we may use the form $u(c) = (c^{1-\sigma})/(1-\sigma)$, $0 < \sigma < 1$, which meets $u > 0$ when $c > 0$. Nevertheless, our main results remain unchanged.

[14] If the fraction of investment constrained by cash is lower, we have found a higher optimal growth rate of money and if the fraction of investment constrained by cash is higher, we have found a lower optimal growth rate of money.

[15] In our quantitative exercises, we find that there is only one negative eigenvalue and thus there is a saddle path toward the steady state.

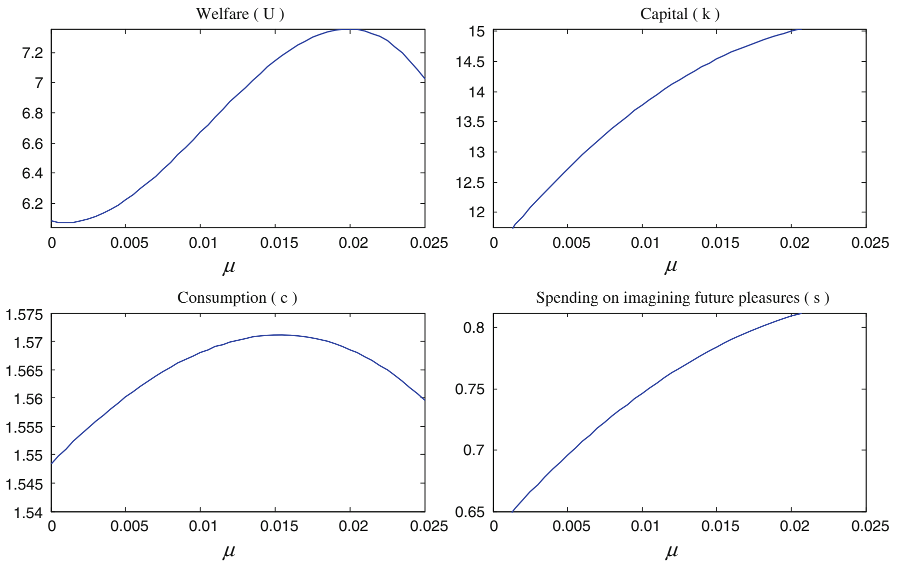

Fig. 2 The optimal growth rate of money

level declines.[16] Thus, the optimal monetary growth rate and thereafter the optimal inflation rate is at $\pi^* = 2.09\%$.[17] Our results suggest that a switch of monetary policies from a 0% inflation rate to a 2.09% inflation rate per annum increases a welfare gain at 6.96% in terms of consumption equivalence per annum.

4 Concluding remarks

This paper integrates the model by Wang and Yip (1992) and the model by Stern (2006) into an otherwise standard optimal growth model with money and resources spent on imagining future pleasures along the lines of Becker and Mulligan (1997). We study the effect of a higher seignorage tax or a higher monetary growth rate on capital, consumption and welfare in the long run.

We find that a higher seignorage tax increases capital and consumption when the fraction of investment constrained by cash is smaller than a threshold. As a result, the Friedman rule of optimum quantity of money does not hold and the optimal inflation rate is positive. Our calibration exercises indicate that the optimal inflation rate is 2.09%. We find that a switch of monetary policies from a zero-percent inflation rate to the optimal inflation rate generates a sizable welfare gain at 6.96% in terms of consumption equivalence per annum.

Finally, there are extensions of our model and we briefly mention two cases. First, Wang and Yip (1992) allowed for a tradeoff between labor and leisure and found a

[16] We find $\varphi_1 = 6.7372$, $\varphi_2 = 0.41$ and $\varphi_3 = 0.3996$ at the optimal monetary growth rate.

[17] The steady state at the optimal monetary growth rate is $k^* = 15.0251$, $c^* = 1.5682$, $s^* = 0.8118$, $y^* = 2.3971$ and $U^* = 7.3474$.

negative relationship between money and capital in the long run with an ambiguous net effect on leisure due to a positive effect in the production side and a negative effect in the utility side. We may extend our model to consider leisure and thus a tradeoff between labor and leisure and reexamine the relationship between money and capital in the long run. Second, we may extend our model to allow for a fraction of spending on imagining future pleasure that is constrained by the cash constraint. It is interesting to reexamine the relationship between money and capital in the long run when a fraction of spending on imagining future pleasure is constrained by the cash constraint.

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5 Appendix

In this section, we prove the stability of our model. Then, we investigate the case when the time preference is along the lines of Uzawa (1968).

5.1 Stability condition

The dynamical system includes (5b), (6b), (6c) and (6e). The dynamical system involves one predetermined state variable. The equilibrium path toward the unique steady state is saddle if there exists only one negative eigenvalue. Taking a linear Taylor's expansion of the dynamic system in the neighborhood of the unique steady state gives

$$\begin{bmatrix} \dot{c}_t \\ \dot{s}_t \\ \dot{k}_t \\ \dot{\lambda}_{mt} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ -1 & -1 & f'(k^*) & 0 \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix} \begin{bmatrix} c_t - c^* \\ s_t - s^* \\ k_t - k^* \\ \lambda_{mt} - \lambda_m^* \end{bmatrix}, \tag{A1}$$

where

$$J_{11} = \frac{1}{u''(c^*)}\left[\left(1-\frac{1}{\varphi}\right)J_{41} + u''(c^*)\rho(s^*)\right],$$

$$J_{12} = \frac{1}{u''(c^*)}\left[\left(1-\frac{1}{\varphi}\right)J_{42} + u'(c^*)\rho'(s^*) - \left(1-\frac{1}{\varphi}\right)\rho'(s^*)\lambda_m^*\right],$$

$$J_{13} = \frac{1}{u''(c^*)}\left\{\left(1-\frac{1}{\varphi}\right)J_{43} - \frac{1}{\varphi}f''(k^*)\lambda_m^*\right\},$$

$$J_{14} = \frac{1}{u''(c^*)}\left\{\left(1-\frac{1}{\varphi}\right)J_{44} - \left(1-\frac{1}{\varphi}\right)\rho(s^*) - \frac{1}{\varphi}f'(k^*)\right\},$$

$$J_{21} = \frac{-1}{\theta^*\rho''(s^*)}\left\{J_{41} - u'(c^*)\rho'(s^*)\right\},$$

$$
\begin{aligned}
J_{22} &= \frac{-1}{\theta^*\rho''(s^*)}[J_{42} - \rho'(s^*)\lambda_m^*] + \rho(s^*),\\
J_{23} &= \frac{-1}{\theta^*\rho''(s^*)} J_{43},\\
J_{24} &= \frac{-1}{\theta^*\rho''(s^*)} \left[J_{44} - \rho(s^*)\right],\\
J_{41} &= \frac{\lambda_m^*}{\Lambda^*} \left\{(1-\varphi)\left[\left(1-\frac{1}{\varphi}\right) - \rho(s^*)\right] + \varphi f'(k^*) + \frac{\varphi}{\theta^*\rho''(s^*)}\right.\\
&\quad \left.\times \left[u''(c^*) + u'(c^*)\rho'(s^*)\right]\right\} - u''(c^*),\\
J_{42} &= \frac{\lambda_m^*}{\Lambda^*} \left\{-(1-\varphi)\frac{u'(c^*)\rho'(s^*)}{u''(c^*)} + \varphi f'(k^*) + \varphi\rho(s^*)\right\} + \rho'(s^*)\lambda_m^*,\\
J_{43} &= \frac{\lambda_m^*}{\Lambda^*} \left\{f''(k^*)\lambda_m^* \frac{1}{\varphi}\frac{(1-\varphi)}{u''(c^*)} - \varphi[f'(k^*)]^2\right\},\\
J_{44} &= 1 + \rho(s^*) + \mu + \frac{\lambda_m^*}{\Lambda^*} \left\{\frac{-(1-\varphi)}{u''(c^*)}\left[\left(1-\frac{1}{\varphi}\right)(1+\mu) - \frac{1}{\varphi}f'(k^*)\right]\right.\\
&\quad \left. -\frac{\varphi}{\theta^*\rho''(s^*)}[1+\mu]\right\},\\
\Lambda^* &= c^* + \frac{1-\varphi}{u''(c^*)}\left(1-\frac{1}{\varphi}\right)\lambda_m^* + \frac{\varphi}{\theta^*\rho''(s^*)}\lambda_m^* > 0.
\end{aligned}
$$

The determinant of the Jacobean matrix in (A1) is

$$
\Xi^{\varphi} = \frac{-\lambda_m^* c^*}{\left[c^*\theta^*\rho''(s^*) + \lambda_m^*\right]u''(c^*)}\hat{\Delta}^{\varphi} < 0,
$$

where

$$
\begin{aligned}
\hat{\Delta}^{\varphi} = f''(k^*) &\left\{u(c^*)\rho''(s^*)(1+\rho(s^*)+\mu) - u''(c^*)\rho(s^*) + u(c^*)\rho'(s^*)^2\right.\\
&\left. - u'(c^*)\rho'(s^*)\left[\rho(s^*)+\mu\right]\right\}\\
&- f'(k^*)\rho'(s^*)\left[1+\varphi(2\rho(s^*)+\mu)\right]\left[u''(c^*)\rho(s^*) + u'(c^*)\rho'(s^*)\right] < 0.
\end{aligned}
$$

Under Condition SS, we obtain $\hat{\Delta}^{\varphi} < 0$. There are thus either one or three negative eigenvalues. To guarantee that there is exactly one eigenvalue, denote as ω the corresponding eigenvalue of the Jacobean matrix in (A1). The characteristic polynomial of the Jacobean matrix is

$$
\Gamma^{\varphi}(\omega) = \omega^4 + a_1\omega^3 + a_2\omega^2 + a_3\omega + \Xi^{\varphi}.
$$

According to the Routh–Hurwitz theorem, the number of eigenvalues with positive real parts of the polynomial $\Gamma^{\varphi}(\omega) = 0$ is equal to the number of variations in signs of the following series,

$$\left\{1, a_1, \frac{a_1a_2 - a_3}{a_1}, \frac{a_1a_2a_3 - a_3^2 - \Xi^{\varphi}a_1^2}{a_1a_2 - a_3}, \Xi^{\varphi}\right\}. \tag{A2}$$

One negative eigenvalue is equivalent to three positive eigenvalues which requires the sign in the series in (A2) to change three times. However, it is difficult to analyze the sign of the series because a_1, a_2 and a_3 are complicated. Nevertheless, in the case of $\varphi = 0$, we can pin down the sign of the series. Specifically, under $\varphi = 0$, (6b), (6c), and (6e) become, respectively,

$$\dot{c}_t = \left[1 + \mu - \frac{u'(c_t)}{\lambda_{mt}} + f'(k_t)\right] c_t, \tag{A3a}$$

$$\dot{s}_t = \frac{1}{-\theta_t \rho''(s_t)}\left[\dot{\lambda}_{mt} - \rho(s_t)\lambda_{mt} - u(c_t)\rho'(s_t)\right], \tag{A3b}$$

$$\dot{\lambda}_{mt} = \lambda_{mt}\left[\rho(s_t) - f'(k_t)\right]. \tag{A3c}$$

Then, the linearized dynamic system of (5b) and (A3a)-(A3c) near the unique steady state is

$$\begin{bmatrix} \dot{c}_t \\ \dot{s}_t \\ \dot{k}_t \\ \dot{\lambda}_{mt} \end{bmatrix} = \begin{bmatrix} -\frac{u''(c^*)c^*}{\lambda_m^*} & 0 & f''(k^*)c^* & \frac{u'(c^*)c^*}{(\lambda_m^*)^2} \\ \frac{u'(c^*)\rho'(s^*)}{\theta^*\rho''(s^*)} & \frac{u(c^*)\rho''(s^*)}{\theta^*\rho''(s^*)} & \frac{f''(k^*)\lambda_m^*}{\theta^*\rho''(s^*)} & \frac{\rho(s^*)}{\theta^*\rho''(s^*)} \\ -1 & -1 & f'(k^*) & 0 \\ 0 & \rho'(s^*)\lambda_m^* & -f''(k^*)\lambda_m^* & 0 \end{bmatrix} \begin{bmatrix} c_t - c^* \\ s_t - s^* \\ k_t - k^* \\ \lambda_{mt} - \lambda_m^* \end{bmatrix}, \tag{A4a}$$

The characteristic polynomial of the Jacobean matrix is

$$\Gamma(\omega) = \omega^4 + b_1\omega^3 + b_2\omega^2 + b_3\omega + \Xi,$$

where

$$\begin{aligned} b_1 &= \frac{u''(c^*)c^*}{\lambda_m^*} - f'(k^*) - \rho(s^*) < 0, \\ b_2 &= \frac{f'(k^*)u(c^*)}{\theta^*} - \frac{u(c^*)u''(c^*)c^*}{\theta^*\lambda_m^*} - \frac{f'(k^*)u''(c^*)c^*}{\lambda_m^*} - \frac{\rho(s^*)\rho'(s^*)\lambda_m^*}{\theta^*\rho''(s^*)} > 0, \\ b_3 &= \frac{f'(k^*)u(c^*)u''(c^*)c^*}{\theta^*\lambda_m^*} + \frac{f'(k^*)\rho'(s^*)\rho(s^*)\lambda_m^*}{\theta^*\rho''(s^*)} - \frac{\rho'(s^*)\rho(s^*)u''(c^*)c^*}{\theta^*\rho''(s^*)} \\ &\quad + \frac{(u'(c^*)\rho'(s^*))^2}{\theta^*\rho''(s^*)\lambda_m^*} - \frac{f''(k^*)u'(c^*)c^*}{\lambda_m^*} - \frac{f''(k^*)\rho(s^*)\lambda_m^*}{\theta^*\rho''(s^*)}, \\ \Xi &= \frac{c^*}{\theta^*\rho''(s^*)}\hat{\Delta} < 0, \end{aligned}$$

$$\hat{\Delta} = f''(k^*)\Big[-u'(c^*)\rho'(s^*)(\rho(s^*)+\mu) + u(c^*)\rho''(s^*)(1+\rho(s^*)+\mu)$$
$$-u''(c^*)\rho(s^*) + u(c^*)(\rho'(s^*))^2\Big]$$
$$-f'(k^*)[u'(c^*)(\rho'(s^*))^2(1+\rho(s^*)+\mu) + u''(c^*)\rho'(s^*)\rho(s^*)] < 0.$$

The number of eigenvalues with positive real parts of the polynomial $\Gamma(\omega) = 0$ is equal to the number of variations in signs of the following series,

$$\left\{1, \quad b_1, \quad \frac{b_1 b_2 - b_3}{b_1}, \; \frac{b_1 b_2 b_3 - b_3^2 - \Xi b_1^2}{b_1 b_2 - b_3}, \; \Xi\right\}. \tag{A4b}$$

First, Conditon S assures $\Xi < 0$; thus, there is either one or three negative eigenvalues. Next,

$$b_1 b_2 - b_3 = -\frac{u(c^*)[f'(k^*)]^2}{\theta^*} - \frac{f'(k^*)u(c^*)^2}{\theta^{*2}} - \frac{u(c^*)\left[u''(c^*)c^*\right]^2}{\theta^*\lambda_m^{*2}}$$
$$+\frac{u''(c^*)c^*u(c^*)^2}{\theta^{*2}\lambda_m^*} - f'(k^*)\left[\frac{u''(c^*)c^*}{\lambda_m^*}\right]^2 + \frac{\left[f'(k^*)\right]^2 u''(c^*)c^*}{\lambda_m^*}$$
$$+\frac{f'(k^*)u(c^*)u''(c^*)c^*}{\theta^*\lambda_{\mathrm{m}}^*} + \frac{u(c^*)\rho''(s^*)\rho(s^*)\rho'(s^*)\lambda_m^*}{[\theta^*\rho''(s^*)]^2}$$
$$-\frac{\left[u'(c^*)\rho'(s^*)\right]^2}{\theta^*\rho''(s^*)\lambda_m^*} + \frac{f''(k^*)u'(c^*)c^*}{\lambda_m^*} + \frac{f''(k^*)\rho(s^*)\lambda_m^*}{\theta^*\rho''(s^*)} < 0.$$

This indicates that the third term in (A4b) is positive. Hence, the number of variations in sign of the series in (A4b) is at least three, implying that there are at least three positive eigenvalues. Sine Condition S assures that there is either one or three eigenvalues, the number of negative eigenvalues is one. Therefore, there is a unique saddle path toward the steady state.

5.2 Uzawa time preferences

In the model with time preferences along the lines of Uzawa (1968), the discount factor is $X_t \equiv \exp[-\int_0^t \rho(c_\tau)d\tau]$. In line with the literature of endogenous discounting, we assume that $\rho\prime(c) > 0 > \rho''(c)$. To ensure that the Hamiltonian is strictly concave, it is necessary to impose $u(c) < 0$ (Obstfeld 1990). Note that under $u(c) < 0$, the shadow price of the discounting, denoted by θ, is negative, which automatically assures a positive intertemporal elasticity of substitution. Denote λ_k as the shadow price of capital and λ_m as the shadow price of real balances. The first order conditions are

$$u'(c_t) - \theta_t \rho'(c_t) = \frac{1}{\varphi}\lambda_{kt} + \left(1 - \frac{1}{\varphi}\right)\lambda_{mt}, \tag{A5a}$$

$$\dot{\lambda}_{kt} = \rho(c_t)\lambda_{kt} - f'(k_t)\lambda_{mt}, \tag{A5b}$$

$$\dot{\lambda}_{mt} = \left[\frac{1}{\varphi} + \rho(c_t) + \pi_t\right]\lambda_{mt} - \frac{1}{\varphi}\lambda_{kt}, \tag{A5c}$$

$$\dot{\theta}_t = -u(c_t) + \theta_t \rho(c_t), \tag{A5d}$$

along with the transversality constraints: $\lim_{t\to\infty} \lambda_{kt} X_t k_t = 0$, $\lim_{t\to\infty} \lambda_{mt} X_t m_t = 0$ and $\lim_{t\to\infty} \theta_t X_t = 0$.

The equilibrium conditions include (A5a)–(A5d), along with the money market clearance condition

$$\dot{m}_t = (\mu - \pi_t)m_t, \tag{A6a}$$

and the good market clearance condition

$$\dot{k}_t = f(k_t) - c_t, \tag{A6b}$$

that solve for $c_t, k_t, m_t, \lambda_{kt}, \lambda_{mt}, \theta_t$.

The steady state is attained if $\dot{k}_t = \dot{\lambda}_{kt} = \dot{\lambda}_{mt} = \dot{\theta}_t = \dot{m}_t = 0$ and $c_t = c^*$ for all t. The steady state is characterized by

$$\frac{f'(k^*)}{1 + \varphi[\rho(c^*) + \mu]} = \rho(c^*), \tag{A7a}$$

$$c^* = f(k^*). \tag{A7b}$$

Case 1, the CIA constraint does not bind investment: $\varphi = 0$. In this case, (A7a) is $f'(k^*) = \rho(c^*)$ and the long-run relationship between money and capital is neutral.

Case 2, the CIA constraint binds a fraction of investment: and $\varphi > 0$. In this case, monetary growth, μ, appears in the left-hand side of (A7a) and a higher monetary growth rate reduces the discounted marginal product of capital in (A7a). As a result, capital is unambiguously lower in the long run.

To analyze the stability condition of the steady state, applying the implicit function theorem to (A5a) gives the following relationship

$$\theta_t = \theta(c_t, \lambda_{kt}, \lambda_{mt}), \tag{A8a}$$

in which $\theta_c \equiv \frac{\partial\theta_t}{\partial c_t} = \frac{u''(c_t) - \theta_t \rho''(c_t)}{\rho'(c_t)}$, $\theta_{\lambda_k} \equiv \frac{\partial\theta_t}{\partial\lambda_{kt}} = -\frac{1}{\varphi}\frac{1}{\rho'(c_t)}$, $\theta_{\lambda_m} \equiv \frac{\partial\theta_t}{\partial\lambda_{mt}} = -(1 - \frac{1}{\varphi})\frac{1}{\rho'(c_t)}$.

Then, differentiating (A5a) with respect to time, with the use of (A5b), (A5c) and (A5d), leads to the following modified Keynes-Ramsey rule,

$$\dot{c}_t=\frac{1}{u''(c_t)-\theta_t\rho''(c_t)}\left\{\left(1-\frac{1}{\varphi}\right)\dot{\lambda}_{mt}+\frac{1}{\varphi}\lambda_{kt}-\rho'(c_t)[-u(c_t)+\theta(c_t,\lambda_{kt},\lambda_{mt})\rho(c_t)]\right\}. \tag{A8b}$$

Moreover, if we use the binding CIA constraint $m_t=(1-\varphi)c_t+\varphi f(k_t)$, the inflation rate is determined by

$$\pi_t=\mu-\frac{(1-\varphi)\dot{c}_t+\varphi f'(k_t)\dot{k}_t}{(1-\varphi)c_t+\varphi f(k_t)}=\pi\left(c_t,\lambda_{kt},k_t,\lambda_{mt}\right). \tag{A8c}$$

Then, the linearized dynamic system of (A5b), (A5c), (A6b) and (A8b) near the unique steady state is

$$\begin{bmatrix}\dot{c}_t\\ \dot{\lambda}_{kt}\\ \dot{k}_t\\ \dot{\lambda}_{mt}\end{bmatrix}=\begin{bmatrix}J_{11} & J_{12} & J_{13} & J_{14}\\ J_{21} & \rho(c^*) & -f''(k^*)\lambda_m^* & -f'(k^*)\\ -1 & 0 & f'(k^*) & 0\\ J_{41} & J_{42} & J_{43} & J_{44}\end{bmatrix}\begin{bmatrix}c_t-c^*\\ \lambda_{kt}-\lambda_k^*\\ k_t-k^*\\ \lambda_{mt}-\lambda_m^*\end{bmatrix}, \tag{A9}$$

where

$$\begin{aligned}
J_{11}&=\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\left(1-\frac{1}{\varphi}\right)J_{41}+\rho(c^*),\\
J_{12}&=\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\left(1-\frac{1}{\varphi}\right)J_{42},\\
J_{13}&=\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\left(1-\frac{1}{\varphi}\right)J_{43},\\
J_{14}&=\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\left\{\left(1-\frac{1}{\varphi}\right)J_{44}-\left(1-\frac{1}{\varphi}\right)\rho(c^*)-\frac{1}{\varphi}f'(k^*)\right\},\\
J_{21}&=\frac{\rho'(c^*)f'(k^*)\lambda_m^*}{\rho(c^*)},\\
J_{41}&=\rho'(c^*)\lambda_m^*+\frac{\lambda_m^*}{\Lambda_U}\left\{(1-\varphi)\left[\mu-\rho(c^*)\right]+\varphi f'(k^*)\right\},\\
J_{42}&=-\frac{1}{\varphi}-\left(\frac{\varphi-1}{\varphi}\right)^2\frac{\lambda_m^*}{\Lambda_U},\frac{1}{u''(c^*)-\theta^*\rho''(c^*)},\\
J_{43}&=\frac{\lambda_m^*}{\Lambda_U}\left\{\frac{1-\varphi}{\varphi}f''(k^*)\lambda_m^*\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}-\varphi f'(k^*)^2+\varphi\mu f'(k^*)\right\},\\
J_{44}&=\frac{1}{\varphi}+\rho(c^*)+\mu+\frac{\lambda_m^*}{\Lambda_U}\left\{\frac{1-\varphi}{\varphi}\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\left[f'(k^*)-\left(1-\frac{1}{\varphi}\right)\right]\right\},\\
\Lambda_U&=c^*-\frac{(1-\varphi)^2}{\varphi}\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\lambda_m^*>0.
\end{aligned}$$

Note that the determinant of the Jacobean matrix in (A9) is

$$\begin{aligned}\Xi_U^{\varphi} &= \frac{1-\varphi}{\varphi}\frac{\lambda_m^*}{\Lambda_U}\frac{f'(k^*)\rho(c^*)}{u''(c^*)-\theta^*\rho''(c^*)}\left[f'(k^*)\rho(c^*)-\frac{1-\varphi}{\varphi}(f'(k^*)-\rho(c^*))\right]\\
&\quad -\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\left[\frac{1}{\varphi}(f'(k^*)-\rho(c^*))+\rho(c^*)\right]\\
&\quad \times\left\{\frac{1}{\varphi}\frac{f''(k^*)\lambda_m^*(1-\varphi)}{u''(c^*)-\theta^*\rho''(c^*)}\frac{\lambda_m^*}{\Lambda_U}\left[\frac{u''(c^*)-\theta^*\rho''(c^*)}{1-\varphi}\frac{\Lambda_U}{\lambda_m^*}-1-\rho(c^*)\right]\right.\\
&\quad \left.-\frac{\varphi\mu\lambda_m^*\rho(c^*)f'(k^*)}{\Lambda_U}\right\}-\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\left[\frac{1}{\varphi}(f'(k^*)-\rho(c^*))+\rho(c^*)\right]\\
&\quad \times\left[\frac{1-\varphi}{\varphi}\frac{\lambda_m^*}{\Lambda_U}\frac{f'(k^*)^2\rho'(c^*)\lambda_m^*}{\rho(c^*)}\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\right.\\
&\quad \left.-\varphi\frac{\lambda_m^*}{\Lambda_U}f'(k^*)(\mu\rho(c^*)+f'(k^*)\rho'(c^*))-(1-\varphi)\frac{\lambda_m^*}{\Lambda_U}f'(k^*)\rho'(c^*)(\rho(c^*)+\mu)\right]\\
&< 0.\end{aligned}$$

Thus, there are either one or three negative eigenvalues. The characteristic polynomial of the Jacobean matrix (A9) is

$$\Gamma_U^{\varphi}(\omega)=\omega^4+d_1\omega^3+d_2\omega^2+d_3\omega+\Xi_U^{\varphi},$$

where

$$\begin{aligned}d_1 &= -\left\{\frac{1}{\varphi}+\rho(c^*)+\mu+\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\right.\\
&\quad \left.\times\left(1-\frac{1}{\varphi}\right)\left[\rho'(c^*)\lambda_m^*-\frac{\lambda_m^*}{\Lambda_U}(1-\varphi)\left(\mu-\rho(c^*)-f'(k^*)-\frac{1}{\varphi}\right)\right]\right\}<0,\end{aligned}$$

$$\begin{aligned}d_2 &= \frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\left[f'(k^*)+2\rho(c^*)\left(1-\frac{1}{\varphi}\right)\right]\left\{\rho'(c^*)\lambda_m^*\right.\\
&\quad \left.+\frac{\lambda_m^*}{\Lambda_U^{\varphi}}\left[(1-\varphi)(\mu-\rho(c^*))+\varphi f'(k^*)\right]\right\}-\frac{1-\varphi}{u''(c^*)-\theta^*\rho''(c^*)}\frac{\lambda_m^*}{\Lambda_U^{\varphi}}\\
&\quad \times\left[f'(k^*)\left(\mu-f'(k^*)\right)+\frac{1-\varphi}{\varphi^2}\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}f''(k^*)\lambda_m^*\right]\\
&\quad -f'(k^*)\left[1+\frac{1-\varphi}{\varphi}\frac{\rho'(c^*)\lambda_m^*}{(u''(c^*)-\theta^*\rho''(c^*))\rho(c^*)}\right]\\
&\quad \times\left[\frac{1}{\varphi}+\left(1-\frac{1}{\varphi}\right)^2\frac{1}{u''(c^*)-\theta^*\rho''(c^*)}\frac{\lambda_m^*}{\Lambda_U^{\varphi}}\right]+\left(f'(k^*)+2\rho(c^*)\right)\end{aligned}$$

$$\times \left[\frac{1}{\varphi} + \rho(c^*) + \mu + \frac{\lambda_m^*}{\Lambda_U^\varphi} \frac{1}{\varphi} \frac{1-\varphi}{u''(c^*) - \theta^* \rho''(c^*)} \left(f'(k^*) - 1 + \frac{1}{\varphi} \right) \right],$$

$$d_3 = \frac{1}{u''(c^*) - \theta^* \rho''(c^*)} \left(1 - \frac{1}{\varphi}\right) \left\{ \rho'(c^*)\lambda_m^* + \frac{\lambda_m^*}{\Lambda_U^\varphi} \left[(1-\varphi)(\mu - \rho(c^*)) + \varphi f'(k^*) \right] \right\}$$

$$\times \left\{ f'(k^*)\rho(c^*) - (f'(k^*) + \rho(c^*)) \left[\frac{1}{\varphi} + \rho(c^*) + \mu \right.\right.$$

$$\left.\left. + \frac{\lambda_m^*}{\Lambda_U^\varphi} \frac{1}{\varphi} \frac{1-\varphi}{u''(c^*) - \theta^* \rho''(c^*)} \left(f'(k^*) - 1 + \frac{1}{\varphi} \right) \right] \right\}$$

$$+ \frac{f'(k^*)}{u''(c^*) - \theta^* \rho''(c^*)} \frac{\lambda_m^*}{\Lambda_U^\varphi} \left[f'(k^*) \left(\mu - f'(k^*) \right) \right.$$

$$\left. + \frac{1-\varphi}{u''(c^*) - \theta^* \rho''(c^*)} \frac{1}{\varphi^2} f''(k^*)\lambda_m^* \right] + (\rho(c^*))^2$$

$$\times \left[\frac{1}{\varphi} + \rho(c^*) + \mu + \frac{\lambda_m^*}{\Lambda_U^\varphi} \frac{1}{\varphi} \frac{1-\varphi}{u''(c^*) - \theta^* \rho''(c^*)} \left(f'(k^*) - 1 + \frac{1}{\varphi} \right) \right]$$

$$- \left[\frac{\rho'(c^*)\lambda_m^* \left(f'(k^*) \right)^2}{(u''(c^*) - \theta^* \rho''(c^*))\rho(c^*)} - \left(f'(k^*) \right)^2 - f'(k^*)\rho(c^*) \right.$$

$$\left. + \frac{\lambda_m^*}{u''(c^*) - \theta^* \rho''(c^*)} \left(1 - \frac{1}{\varphi}\right) \left(f'(k^*)\rho(c^*) - f''(k^*) \right) \right]$$

$$\times \left[\frac{1}{\varphi} + \frac{1}{u''(c^*) - \theta^* \rho''(c^*)} \frac{\lambda_m^*}{\Lambda_U^\varphi} \left(1 - \frac{1}{\varphi}\right)^2 \right].$$

The economy exhibits the saddle-path property if $d_3 > d_1 d_2$. However, it is difficult to sign $d_1 d_2 - d_3$ analytically because d_2 and d_3 are complicated. Nevertheless, we can show that in the case of $\varphi = 0$ the dynamic system is saddle-point stable. To see this, we manipulate the optimality conditions and the binding CIA constraint, (A5b) and (A8b) become, respectively,

$$\frac{\dot{\lambda}_{mt}}{\lambda_{mt}} = -f'(k_t) + \rho(c_t), \tag{A10a}$$

$$\frac{\dot{c}_t}{c_t} = \mu + 1 + f'(k_t) + [u'(c_t) - \theta_t \rho'(c_t)] \frac{1}{\lambda_{mt}}. \tag{A10b}$$

The dynamic system consists of (A5d), (A6b), (A10a) and (A10b). Linearizing these equations around the steady state gives

$$\begin{bmatrix} \dot{c}_t \\ \dot{\lambda}_{mt} \\ \dot{k}_t \\ \dot{\theta}_t \end{bmatrix} = \begin{bmatrix} \frac{-(u''(c^*)-\theta^*\rho''(c^*))c^*}{\lambda_m^*} & \frac{(u'(c^*)-\theta^*\rho'(c^*))c^*}{\lambda_m^{*2}} & f''(k^*)c^* & \frac{\rho'(c^*)c^*}{\lambda_m^*} \\ \rho'(c^*)\lambda_m^* & 0 & -f''(k^*)\lambda_m^* & \rho(c^*) \\ -1 & 0 & f'(k^*) & 0 \\ -u'(c^*)+\theta^*\rho'(c^*) & 0 & 0 & \rho(c^*) \end{bmatrix} \times \begin{bmatrix} c_t - c^* \\ \lambda_{mt} - \lambda_m^* \\ k_t - k^* \\ \theta_t - \theta^* \end{bmatrix}.$$

The characteristic polynomial of the Jacobean matrix is

$$\Gamma_U(\omega) = \omega^4 + e_1\omega^3 + e_2\omega^2 + e_3\omega + \Xi_U,$$

where

$$\begin{aligned} e_1 &= -f'(k^*) - \rho(c^*) + \frac{u''(c^*) - \theta^*\rho''(c^*)}{\lambda_m^*}c^* < 0, \\ e_2 &= f'(k^*)\rho(c^*) - \frac{u''(c^*) - \theta^*\rho''(c^*)}{\lambda_m^*}c^*(f'(k^*) + \rho(c^*)) - f''(k^*)c^* > 0, \\ e_3 &= \frac{u''(c^*) - \theta^*\rho''(c^*)}{\lambda_m^*}f'(k^*)\rho(c^*) + \frac{u'(c^*) - \theta^*\rho'(c^*)}{\lambda_m^*} \\ &\quad \times c^*[f'(k^*)\rho'(c^*) - f''(k^*)] + f''(k^*)\rho(c^*)c^*, \\ \Xi_U &= \frac{u'(c^*) - \theta^*\rho'(c^*)}{\lambda_m^*}c^*\rho(c^*)\left[f''(k^*) - f'(k^*)\rho'(c^*)\right] < 0. \end{aligned}$$

The determinant Ξ_U <0 indicates that the numbers of negative eigenvalue are either three or one. Furthermore, the number of eigenvalues with positive real parts of the polynomial $\Gamma_U(\omega) = 0$ is equal to the number of variations in signs of the following series,

$$\left\{1, e_1, \frac{e_1e_2 - e_3}{e_1}, \frac{e_1e_2e_3 - e_3^2 - \Xi_U e_1^2}{e_1e_2 - e_3}, \Xi_U\right\}.$$

As $e_1e_2 - e_3 < 0$, there are at least three positive eigenvalues. Since $\Xi_U < 0$ indicates that there are either one or three negative eigenvalues, the number of negative eigenvalues is one. Thus, the steady state is a saddle point.

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