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Congestible public goods and local indeterminacy: A two-sector endogenous growth model

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Abstract

This paper shows that the congestible public goods can generate local indeterminacy in a two-sector, constant-return human capital enhanced growth model. While the productive public good exerts positive sector-specific externalities, the congestion effect generates negative aggregate externalities. The sector-specific externalities alone arising from productive public goods cannot establish local indeterminacy without the combination of negative externality in a model with social constant return technology. Congestible public good generates local indeterminacy if the degree of productive public good externality and the degree of congestion effect are large enough. The condition for indeterminacy is independent of the factor intensity rankings. The conditions are quantitatively assessed and the required parameter values for the degrees of public good externality and congestion are consistent with the estimated values in existing literature.

JEL classification: O40; O41

Keywords: Two-sector model; Congestible public good; Indeterminacy

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1. Introduction

Several economic growth models have recently established the existence of indeterminate equilibrium paths. The indeterminacy of equilibrium paths is significant as it lays the groundwork for endogenous growth fluctuations that provide for an explanation for large, persistent variations in growth experiences, where different countries follow different equilibrium trajectories toward a balanced growth path. The purpose of this paper is to show that the congestible public goods can establish local indeterminacy in a two-sector, human capital enhanced growth model with constant return. The mechanism differs from the two mechanisms in existing literature.

One of the existing mechanisms is the sector-specific externalities. Benhabib et al. (2000) and Mino (2001) belong to this line of thought and establish the existence of indeterminacy. The other mechanism is distortionary factor taxation. Bond et al. (1996) limit the government activity to the collection of taxes and payment of transfers, and find that the balanced growth path is locally indeterminate if the factor tax rates are very different both across sectors and across factors. Raurich (2001) extends Bond et al. (1996) and shows that with the consideration of unproductive public spending, indeterminacy is empirically plausible. Finally, Ben-Gad (2003) combines both the distortionary factor taxation and the sector-specific externalities in the model with unproductive government spending, and finds that indeterminacy is not possible when either one of the two mechanisms is added to the model in isolation. With elastic labor supply and physical capital employed in both sectors, indeterminacy emerges for varying combinations of factor taxation and sector-specific externalities.

In this paper, we construct a two-sector, constant-return growth model with productive public services in the goods sector financed by the income taxation in the goods sector,[1] and the use of productive public services is subject to a congestion effect in association with the aggregate activities in the economy.[2] The aggregate technology in the goods sector exhibits positive sector-specific externalities through productive public good in association with sector physical and human capital and negative aggregate externalities via congestion effects in association with aggregate activities. The sector-specific externalities alone here cannot establish local indeterminacy without the combination of negative externality.

We find that congestible productive pubic goods establish local indeterminacy if the degrees of public good externality and congestion effects are sufficiently large.

[1]Several studies have documented the productive effects of public goods on production. Aschauer (1989) has documented positive impacts of public expenditure on the productivity of private capital using the data from the US, and Lynde and Richmond (1993a) have found an essential role of public capital in enhancing the productivity growth of UK manufacturing. See Gramlich (1994) for a survey of this empirical literature, most of which is for the United States.

[2]Congestion in the use of public goods has received much attention in spatial economics (e.g., Edwards, 1990; Batten and Karlsson, 1996) and growth models (e.g., Barro and Sala-i-Martin, 1992; Turnovsky, 1996; Glomm and Ravikumar, 1997). Even national defense, sometimes cited as the purest of public goods, is not congestion free.

Intuitively, when the representative agent expects a higher return in one sector, it will allocate more physical and human capital from the other sector to that sector. Such reallocation makes the marginal products of physical and human capital in that sector increase as the law of diminishing marginal returns no longer holds from the social perspective under sufficiently large degrees of productive public good externality and congestion effects. Moreover, the conditions for indeterminacy are independent of the factor intensity rankings, unlike existing works that required the goods sector be human capital intensive and the human capital sector be physical capital intensive from the private perspective.

We calibrate the model based on the US economy and simulate the values of the degrees of public goods services and congestion. The parameter values simulated for local indeterminacy are plausible; the required degree of the congestion effect and the degrees of the contribution of productive public goods are consistent with the estimated values in existing empirical studies. See, among others, McMillan (1989) for the former effect and Lynde and Richmond (1993b) and Shioji (2001) for the latter effect.

Our paper contributes to existing works that established local indeterminacy via negative externalities by Kehoe (1991) and Meng and Yip (2005) in one-sector growth models and by Weder (2001) in a two-sector growth model of small open economies. These works all require technologies with *increasing returns* from the social perspective that are not supported by empirical studies (Burnside, 1996; Basu and Fernald, 1997). Negative aggregate externality alone without productive public goods cannot generate indeterminacy in our model unless the goods production is of increasing returns from the private perspectives, an improper setup for a competitive firm. Our result is obtained under technologies with social *constant returns*. Congestible public good that combines both positive and negative externalities is thus a proper mechanism that can establish local indeterminacy with production technologies of social constant returns.

The rest of the paper is organized as follows. While Section 2 sets up the basic model and analyzes the balanced growth path, Section 3 theoretically analyzes the equilibrium dynamics and the conditions for local indeterminacy. In Section 4 we conduct numerical simulations based on the US economy. Finally, concluding remarks are made in Section 5.

2. The model

2.1. The environment

The model is based upon Barro (1990), Rebelo (1991) and Bond et al. (1996). The economy is comprised of a representative agent with two production sectors, referred to as the goods Sector, Y, and the education Sector, X. Human and physical capital are both necessary inputs in each sector, with Sector Y enhanced by congestible public goods.[3]

[3]We have shown that the results are the same if both sectors are affected by congestible public goods, but it is simpler to model one sector affected by congestible public goods.

Specifically, the technology in both sectors is of constant returns and, to simplify, is of a Cobb–Douglas form:

$$y = (vk)^{\alpha}(uh)^{\beta}G_y^{\gamma}, \;\; k(0)>0 \text{ and } h(0)>0 \text{ given}, \tag{1a}$$

$$x = A[(1-v)k]^{\eta}[(1-u)h]^{1-\eta}, \tag{1b}$$

in which y and x are individual output in the goods sector and the education sector, respectively, $0<v<1$ and $0<u<1$ are the fractions of physical, k, and human capital, h, respectively, allocated to the goods sector. Parameter $A>0$ is productivity coefficient in Sector X; $0<\alpha<1$ (*resp.* $0<\eta<1$) and $0<\beta<1$ (*resp.* $1-\eta$) are the shares of physical and human capital in the goods (*resp.* education) sector. In addition, G_y is the perceived amount of public goods services received by the representative agent in the goods sector, with parameter γ representing the degree to which the public services affect the productivity.

The perceived amount of public goods services received by the representative agent is given in the following form[4]:

$$G_y = \phi G\left(\frac{1}{H}\right)^{\psi}, \tag{2}$$

where G is aggregate public spending with the fraction $0<\phi<1$ allocated to productive services, and the services received are decreasing in the aggregate activities/usages. Parameter ψ measures the degree of congestion: if $\psi=0$, the public service is non-rival and non-excludable and is therefore a pure public good, and if $\psi=1$, the congestion is directly proportional to the aggregate activities in the economy as in Barro and Sala-i-Martin (1992) and Glomm and Ravikumar (1994). While the aggregate activities may be represented by aggregate human capital H, the results are independent of the way aggregate activities are formalized.[5]

Let us make some comments on the formulation in (2). The congestion effect associated with aggregate activities is represented by aggregate inputs, which is in line with existing literature (e.g., Glomm and Ravikumar, 1997). The choice of aggregate human capital to stand for aggregate activities in (2) is for the simplicity of presentation while keeping the keys for local indeterminacy. Alternatively, the same results can be obtained if the congestion effect is represented by aggregate physical capital, or represented by both aggregate physical and human capitals. However, the algebra becomes more complicated. See Appendix B and C.[6] We must mention that the congestion effect needs to grow in consistence with the economy and thus cannot be represented by number of workers without intangible knowledge embedded. We also remark that the congestion effect cannot be represented by output Y to a power

[4]The parametric form follows from Turnovsky (1996) and Glomm and Ravikumar (1997), which indicates that an individual's usage of the public services is congested by the aggregate usage making the congestion effect to increase in the absolute size of the economy.

[5]If we use $G_y = \phi G/(HK)^{\psi}$ in place of (2), the results in the paper are the same.

[6]To save space, we sketch the basic results in the appendices. Full appendices are available from the authors upon request.

less than 1 as G is linear in Y and will make the congestion effect to vanish. Then, one of the keys leading to local indeterminacy in this paper no longer works.

Public spending is financed by income taxes in the goods sector, with a flat tax rate, $\tau > 0$, as in Barro (1990) and many others. For simplicity, we also assume there is no depreciation for the stocks of physical and human capital. As a result, the motions of the two kinds of capital stock for the representative agent are given by

$$\dot{k} = (1-\tau)y - c, \tag{3a}$$

$$\dot{h} = x, \tag{3b}$$

where c is individual consumption. We have used lower cases c, y, x, k and h to denote individual levels. In what follows, their corresponding upper cases C, Y, X, K and H are used to denote the corresponding aggregate value.

We do not allow for the depreciation rate for the laws of motions of physical and human capitals in (3a) and (3b). In endogenous growth models, it is known that the results of indeterminacy may be sensitive to the depreciation rates. In Appendix D we allow for the depreciation rates. We have shown that if the depreciation rates are the same for physical and human capital, the results are the same. If the depreciation rates are different, the results are the same if the rate of time preference rate is small. When the depreciation rates are different and the rate of time preference rate is not small, local indeterminacy requires sufficiently large degrees of public sending externalities and congestion effects. We have calibrated this model and conducted simulation (in Appendix Table D1). We find the degrees of public sending externalities and congestion effects required for local indeterminacy are plausible. Therefore, the presence of the depreciation rates does not affect our results.

The preference of the representative agent is a discounted lifetime utility, with a felicity exhibiting a constant, intertemporal elasticity of substitution:

$$U = \int_0^\infty \mathrm{e}^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t, \tag{4}$$

in which $\rho > 0$ is the instantaneous time-preference rate, and σ is the reciprocal of the intertemporal elasticity of substitution.

Finally, to complete the model, the government budget constraints need to balance:

$$G = \tau Y. \tag{5}$$

Together (2) and (5) the aggregate technology for individuals in Sector Y is

$$y = (\phi\tau)^{\gamma/(1-\gamma)} (vk)^{\alpha} (uh)^{\beta} (vK)^{\alpha\gamma/(1-\gamma)} (uH)^{\beta\gamma/(1-\gamma)} H^{-\psi\gamma/(1-\gamma)}. \tag{6}$$

Examining (6), there are negative aggregate externalities via the congestion effect in association with aggregate activities and positive sector-specific externalities through public services in association with sector physical and human capital. The degree of negative aggregate externality is $\xi \equiv \psi\gamma/(1-\gamma)$, whereas the degree of positive sector-specific externality associated with vK and uH is $\delta \equiv \alpha\gamma/(1-\gamma)$ and $\varepsilon \equiv \beta\gamma/(1-\gamma)$, respectively. A special feature is that the sector-specific and the negative externalities are all related to $\gamma/(1-\gamma)$, a share increasing in the degree of the public good externalities. As will be shown, both the degree of public good externalities and the degree of congestion effects play important in economic fluctuations.

The following parametric restrictions are imposed.

Assumption 1.

(i) $\beta - \gamma\psi > 0;$

(ii) $$\alpha + \beta + \frac{(\alpha + \beta - \psi)\gamma}{1 - \gamma} = 1;$$

(iii) $0 \leqslant \psi \leqslant 1.$

While Assumption 1(i) assures that human capital are productive in Sector Y in an aggregate economy, Assumption 1(ii) guarantees technologies with constant returns to scale with respect to all growing inputs, i.e., K, H and G and that implies $\alpha + (\beta - \gamma\psi) = 1 - \gamma$. Finally, in Assumption 1(iii), $\psi \leqslant 1$ is made to ensure the dominance of the effects of public spending services over the congestion effect so that $\alpha + \beta \leqslant 1$ and thus avoids private increasing returns to scale.

Let us discuss some special cases of our model in association with (6). When there are no sector-specific externalities and only aggregate externalities from social human capital and the education sector requires only the input of human capital; i.e., if $\delta = \varepsilon = 0$, $\xi > 0$, $\eta = 0$ and $\alpha + \beta = 1$, the model becomes the standard Lucas' (1988) model. As is well known, the balanced-growth path in this case is locally determinate.[7]

Alternatively, if there are no aggregate externalities and the education sector requires both inputs of physical and human capital; i.e., if $\xi = 0$ and $0 < \eta < 1$, the model is reduced to the special case of Benhabib et al. (2000) and Mino (2001) in which there is no sector-specific externalities in the education sector.

The present model implicitly assumes $\alpha/\beta = \delta/\varepsilon$, and hence there is no divergence in factor-intensity ranking between the private and social technologies. This indicates that the required conditions of a divergence in factor-intensity ranking between the private and social technologies in order to establish local indeterminacy in Benhabib et al. (2000) and Mino (2001) cannot hold. As our analysis below shows, the keys for local indeterminacy in the present study are (i) the existence of aggregate, negative externalities generated by human capital in the final goods sector and (ii) the use of physical capital in the production of new human capital. These factors lead to divergences in the relative marginal products of capital between the private and social technologies that are required for indeterminacy.

[7] Benhabib and Perli (1994, Proposition 2, pp.123–25), have shown that it is possible to exhibit local indeterminacy in the standard Lucas model, but it requires too high an intertemporal elasticity of substitution.

2.2. The optimization and equilibrium

Given the tax rates, public spending and initial $k(0)$ and $h(0)$, the representative agent's problem is to choose c, v, u, k and h, in order to maximize its discounted lifetime utility in (4), subject to constraints in (1a,b), (2), and (3a,b). Denote λ and μ as the co-state variables associated with k and h, respectively. Then, the necessary conditions are:

$$c^{-\sigma} = \lambda, \tag{7a}$$

$$\lambda(1-\tau)\alpha\frac{y}{v} = \mu\eta\frac{x}{1-v}, \tag{7b}$$

$$\lambda(1-\tau)\beta\frac{y}{u} = \mu(1-\eta)\frac{x}{1-u}, \tag{7c}$$

$$\lambda(1-\tau)\alpha\frac{y}{k} + \mu\eta\frac{x}{k} = \rho\lambda - \dot{\lambda}, \tag{7d}$$

$$\lambda(1-\tau)\beta\frac{y}{h} + \mu(1-\eta)\frac{x}{h} = \rho\mu - \dot{\mu}, \tag{7e}$$

together transversality conditions $\lim_{t\to\infty} e^{-\rho t}\lambda k = 0$ and $\lim_{t\to\infty} e^{-\rho t}\mu h = 0$.

Condition (7a) equates the marginal utility of consumption to the marginal cost, the shadow price of physical capital, while (7b) and (7c) equate the marginal product of physical capital and human capital between the goods and the education sector. Finally, (7d) and (7e) are the Euler equations governing the optimal accumulation for physical and human capital, respectively.

From the Pareto complements in physical and human capital in the technology, we obtain:

Proposition 1. *The fraction of human capital and the fraction of physical capital employed in the final goods sector are positively related.*

Proof. If we divide (7b) by (7c), we obtain

$$v = \frac{\alpha(1-\eta)u}{\alpha(1-\eta)u + \eta\beta(1-u)} \equiv v(u) < 1 \text{ if } u < 1, \tag{8}$$

which leads to $v'(u) > 0$.

Definition. Given an income tax rate τ and initial physical and human capital $K(0)$ and $H(0)$, a perfect foresight equilibrium (PFE) is a tuple $\{Y/H, X/H, G/H, G_y/H, v, u, \dot{K}/K, \dot{H}/H, \dot{C}/C, \lambda, \mu\}$, with $h = H$, $k = K$ and $c = C$, that satisfies:

(i) production technologies, (1a,b), (2);
(ii) household budget constraint and law of motions, (3a,b);

(iii) household optimization, (7a)–(7e), together with the two transversality conditions;
(iv) government budget constraints. (5).

To analyze the PFE, it is necessary to transform the system with non-stationary variables into a system with stationary variables. Following Bond, et al. (1996) we transform the economic system into the structure with variables $\{p, s, z\}$, where $p \equiv \mu/\lambda$, $s \equiv C/H$, and $z \equiv K/H$. In the following we briefly describe the transformation, with detailed derivation found in the Appendix A.

First, if we utilize (1a, b), (2) and (5), we rewrite (7b) as

$$\Phi_1 u^{\psi\gamma/1-\gamma}\left(\frac{v(u)}{u}\right)^{\Lambda} = z^{-\Lambda}p, \tag{9a}$$

where

$$\Phi_1 = \frac{\alpha^2(1-\tau)}{\eta(1-\eta)A}\left[\frac{\eta\beta}{\alpha(1-\eta)}\right]^{1-\eta}(\phi\tau)^{\gamma/1-\gamma}$$

and

$$\Lambda = \frac{\alpha(1-\eta)-\eta(\beta-\gamma\psi)}{1-\gamma} = \left(\alpha+\frac{\alpha\gamma}{1-\gamma}\right)(1-\eta)-\eta\left(\beta+\frac{\beta\gamma}{1-\gamma}-\frac{\gamma\psi}{1-\gamma}\right),$$

which using relationship in (8) is rewritten as

$$u = u(p, z), \tag{9b}$$

with

$$\frac{\partial u}{\partial p} = \frac{-\alpha(1-\eta)u}{p\Gamma} \gtrless 0 \text{ if } \Gamma \lessgtr 0,$$

$$\frac{\partial u}{\partial z} = \frac{\alpha(1-\eta)u\Lambda}{z\Gamma}$$

and

$$\Gamma = \Lambda[\alpha(1-\eta)-\eta\beta]v - \frac{\gamma\psi}{1-\gamma}\alpha(1-\eta).$$

Next, while (7b) and (7d) lead to an expression for $\dot{\lambda}/\lambda$, (7c) and (7e) yield an expression for $\dot{\mu}/\mu$, and (7a) and (7d) lead the following relationship:

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}[(1-\tau)r(u(p,z),p)-\rho], \tag{10}$$

where $r(u(p,z),p) = \Phi_2 p^{-(\beta-\gamma\psi)/\Lambda(1-\gamma)}u(p,z)^{\gamma\psi(1-\eta)/\Lambda(1-\gamma)} \gtrless 0$, and $\Phi_2 = \alpha(\phi\tau)^{\gamma/(1-\gamma)}\Phi_1^{\beta-\gamma\psi/\Lambda(1-\gamma)}$.

Finally, (3a), together (7b,d) and (5), yields an expression for $\dot{K}/K$, whereas (3b), along with (7c,e) and (5), leads to an expression for $\dot{H}/H$.

Therefore, we transform the equilibrium conditions into a stationary system as follows:

$$\frac{\dot{p}}{p}=\frac{\dot{\mu}}{\mu}-\frac{\dot{\lambda}}{\lambda}=(1-\tau)r(u(p,z),p)-w(u(p,z),p), \tag{11a}$$

$$\frac{\dot{s}}{s}=\frac{\dot{C}}{C}-\frac{\dot{H}}{H}=\frac{(1-\tau)r(u(p,z),p)-\rho}{\sigma}-\frac{1-u(p,z)}{1-\eta}w(u(p,z),p), \tag{11b}$$

$$\frac{\dot{z}}{z}=\frac{\dot{K}}{K}-\frac{\dot{H}}{H}=\frac{v(u(p,z))}{\alpha}(1-\tau)r(u(p,z),p)-\frac{s}{z}-\frac{1-u(p,z)}{1-\eta}w(u(p,z),p), \tag{11c}$$

where

$$w(u(p,z),p)=\Phi_3 p^{\eta/\Lambda}u(p,z)^{-\gamma\psi\eta/\Lambda(1-\gamma)}\geqslant 0 \text{ and } \Phi_3=(1-\eta)\left[\frac{\eta\beta}{\alpha(1-\eta)}\right]^{\eta}\Phi_1^{-\eta/\Lambda}.$$

System (11a)–(11c) determines p, s and z in a BGP. We then determine u and v using (9b) and (8), respectively. Ratios, Y/H, X/H, G/H and G_y/H are determined using (1a), (1b), (5) and (2). Finally, growth rates, $\dot{C}/C$, $\dot{K}/K$, $\dot{H}/H$, $\dot{\lambda}/\lambda$, and $\dot{\mu}/\mu$ are determined from (7a), (3a), (3b), (7d) and (7e).

2.3. Balanced growth path

We now analyze the equilibrium in a steady state. A steady state is a PFE with a *balanced growth path* (BGP) under which great ratios p, s and z and fractions u and v are constant, and thus the quantity variables grow at a constant rate. As a result, $\dot{p}/p=\dot{s}/s=\dot{z}/z=0$ along a BGP. To determine a BGP, we substitute (11a) into (11b) to obtain

$$\Phi_4\left(\frac{1}{\sigma}-\frac{1-u}{1-\eta}\right)(u)^{\gamma\psi\eta/[\eta(1-\gamma)+\beta-\gamma\psi]}=\frac{\rho}{\sigma}, \tag{12}$$

where

$$\Phi_4=\left\{\left[A(1-\eta)\left[\frac{\eta\beta}{\alpha(1-\eta)}\right]^{\eta}\right]^{\beta-\gamma\psi}[(1-\tau)\alpha]^{\eta(1-\gamma)}(\phi\tau)^{\gamma\eta}\right\}^{1/[\eta(1-\gamma)+\beta-\gamma\psi]}>0.$$

We are now ready to analyze the existence and uniqueness of a non-degenerate BGP with positive growth. We establish the results in four steps.

First, examining (12), the right-hand side is independent of u with a positive, bounded value $0<\rho/\sigma<\infty$, while the left-hand side is a function of u, denoted as $LHS(u)$. It is obvious that the $LHS(u)=0$ at both $u=0$ and $u=u_2\equiv[\sigma-(1-\eta)]/\sigma$, with $u_2\geqslant 0$ if $\sigma\geqslant 1-\eta$, which is the case if $\sigma\geqslant 1$. Moreover, the $LHS(u)<0$ for $0<u<u_2$, and $LHS(u)>0$ for $u>u_2$ and is increasing in u for all $u\geqslant u_2$. Thus, the $LHS(u)$ intersects the right-hand side in (12) at a unique interior $[\sigma-(1-\eta)]/\sigma<u^*<1$ if $LHS(1)=\Phi_4/\sigma>\rho/\sigma$. We impose[8]

[8]The requirement $\sigma\geqslant 1$ is stronger than what is necessary here. Yet, if $\sigma\geqslant 1$, then $u^*>\eta$, a result that will be used when we determine the sign of the bordered Hessian in Section 3 later.

Condition S. $\sigma \geqslant 1$ and $\Phi_4 > \rho$.

Next, substituting u^* into (8) leads to a unique interior v^*, with

$$0 < \frac{\alpha[\sigma - (1-\eta)]}{\alpha[\sigma - (1-\eta)] + \eta\beta} < v^* < 1.$$

Moreover, substituting $0 < u^* < 1$ into $\dot{p} = 0$ in (11a) yields $(1-\tau)r^*(u^*, p) = w^*(u^*, p)$, and determines a unique p^*. Substituting p^* into $\dot{s} = 0$ in (11b) together (9b) yields z^*. In addition, $\dot{z} = 0$ in (11c) together (11b) lead to

$$s^* = z^* \left[(1-\tau)r^* \left(\frac{v^*}{\alpha} - \frac{1}{\sigma} \right) + \frac{\rho}{\sigma} \right].$$

Finally, (11a) and (11b) results in

$$(1-\tau)r^* = \frac{1-\eta}{1-\eta-\sigma(1-u^*)}\rho > \rho,$$

as

$$\frac{\sigma - (1-\eta)}{\sigma} < u^* < 1$$

in a BGP. As a consequence, the economic growth rate in (10) is positive in a BGP. Moreover, the result $(1-\tau)r^* > \rho$ together $v^* > 0$ leads to

$$s^* > z^* \left[\rho \left(\frac{v^*}{\alpha} - \frac{1}{\sigma} \right) + \frac{\rho}{\sigma} \right] > 0.$$

We summarize the above results in:

Proposition 2. *Under Assumption 1 and Condition S, there exists a unique interior BGP with a positive economic growth rate.*

3. Transitional dynamics and intuition

3.1. Transitional dynamics

To investigate the local dynamics of the economical system, we take a linear Taylor's expansion of (11a–c) in the neighborhood of the unique BGP. The expansion leads to

$$\begin{pmatrix} \dot{p} \\ \dot{s} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} J_{11} & 0 & J_{13} \\ J_{21} & 0 & J_{23} \\ J_{31} & -1 & J_{33} \end{pmatrix} \begin{pmatrix} p - p^* \\ s - s^* \\ z - z^* \end{pmatrix}, \tag{13}$$

where

$$J_{11} = \frac{-w^*}{(1-\gamma)\Lambda\Gamma} \{\gamma\psi\alpha(1-\eta) + \Gamma[\eta(1-\gamma) + \beta - \gamma\psi]\},$$

$$J_{13} = \frac{\alpha(1-\eta)\gamma\psi w^* p^*}{(1-\gamma)\Gamma z^*} \gtrless 0, \text{ if } \Gamma \gtrless 0,$$

$$J_{21} = \frac{-s^* w^*}{\Gamma p^*}\left\{\frac{\gamma\psi\alpha(1-\eta)}{(1-\gamma)\Lambda}\left[\frac{1-\eta}{\sigma} + \frac{\eta(1-u^*)}{1-\eta}\right] + \frac{\Gamma}{(1-\gamma)\Lambda}\left[\frac{\beta-\gamma\psi}{\sigma} + \frac{\eta(1-\gamma)(1-u^*)}{1-\eta}\right] + \alpha u^*\right\},$$

$$J_{23} = \frac{\alpha s^* w^*}{\Gamma z^*}\left\{\frac{(1-\eta)\gamma\psi}{1-\gamma}\left[\frac{1-\eta}{\sigma} + \frac{\eta(1-u^*)}{1-\eta}\right] + \Lambda u^*\right\},$$

$$J_{31} = \frac{-z^* w^*}{\Gamma p^*}\left\{\frac{\gamma\psi\alpha(1-\eta)}{(1-\gamma)\Lambda}\left[\frac{(1-\eta)v^*}{\alpha} + \frac{\eta(1-u^*)}{1-\eta}\right] + \frac{\Gamma}{(1-\gamma)\Lambda} \times\left[\frac{(\beta-\gamma\psi)v^*}{\alpha} + \frac{\eta(1-\gamma)(1-u^*)}{1-\eta}\right] + \alpha u^* + \frac{(1-\eta)v^*(1-v^*)}{1-u^*}\right\},$$

$$J_{33} = \frac{w^*}{\Gamma}\left\{\alpha(1-\eta)\left[\frac{\gamma\psi}{1-\gamma}\left(\frac{(1-\eta)v^*}{\alpha} + \frac{\eta(1-u^*)}{1-\eta}\right) + \Lambda\left(\frac{v^*(1-v^*)}{\alpha(1-u^*)} + \frac{u^*}{1-\eta}\right)\right] + \frac{\Gamma s^*}{z^*}\right\}.$$

The Jacobean matrix in (13), denoted as J, determines the local dynamic properties of the economical system. The 3×3 system includes a state-like variable, whose initial value $z(0)$ is predetermined, and two control-like variables, p and s, which can adjust instantaneously. Therefore, the equilibrium path in the neighborhood of the unique BGP is locally determinate if the Jacobean matrix has only one eigenvalue with negative real parts (stable roots). In contrast, if the Jacobean matrix has two, or a larger number of, eigenvalues with negative real parts, then the equilibrium path near the unique BGP is locally indeterminate.

According to the Routh–Hurwitz theorem, the number of characteristic roots with positive real parts equals the number of variations of signs in $\{-1, TrJ, -BJ + DetJ/TrJ, DetJ\}$, in which TrJ, BJ and $DetJ$ denote the trace, the determinant of the Bordered Hessian, and the determinant of matrix J, respectively. It follows that there is a total of eight possible types of variations in sign. In Table 1, the first four rows exhibit cases in which the number of changes in signs are less than or equal to one and thus the number of eigenvalues with negative real parts is larger than or equal to two. If any of the cases in the first four rows in Table 1 emerges in equilibrium, then the BGP is locally indeterminate.

The determinant of the bordered Hessian of the Jacobean matrix is

$$BJ = \frac{-s^* w^*}{(1-\gamma)\Gamma z^*}\left\{\alpha\gamma\psi\left[(1-\eta)\left(1 - \frac{1-\eta}{\sigma}\right) + \eta(u^* - \eta)\right] + \frac{[\eta(1-\gamma) + (\beta-\gamma\psi)]w^* z^*}{\alpha s^* u^*} M(u^*)\right\}, \tag{14}$$

Table 1
Types of dynamic properties

-1	TrJ^*	$-BJ^*+DetJ^*/TrJ^*$	$DetJ^*$	Number of negative roots	Path toward BGP
$-$	$-$	$-$	$-$	3	Indeterminate
$-$	$+$	$+$	$+$	2	Indeterminate
$-$	$-$	$-$	$+$	2	Indeterminate
$-$	$-$	$+$	$+$	2	Indeterminate
$-$	$+$	$-$	$+$	0	Source
$-$	$+$	$+$	$-$	1	Determinate
$-$	$+$	$-$	$-$	1	Determinate
$-$	$-$	$+$	$-$	1	Determinate

where

$$M(u^*) \equiv \eta\beta v^{*2} + \alpha^2 u^{*2} + \left\{[\alpha(1-\eta)-\eta\beta] - \frac{\alpha[\alpha(1-\eta)-\eta(\beta-\gamma\psi)]u^*}{[\eta(1-\gamma)+\beta-\gamma\psi]v^*}\right\} \times \left[v^* - \frac{\alpha(1-u^*)}{1-\eta}\right] v^* u^*.$$

Condition S implies that $1 > (1-\eta)/\sigma$, $u^* > \eta$ and $v^* > \alpha(1-u^*)/(1-\eta)$ and hence, $M(u^*) > 0$. Consequently, the value of the large braces in BJ is positive. It follows that $BJ >$ (*resp.* $<$)0 if $\Gamma <$ (*resp.* $>$) 0.

Moreover, the determinant of the Jacobean (i.e., the product of the eigenvalues) is

$$DetJ = \frac{-\alpha(1-\eta)u^* s^* w^{*2}}{(1-\gamma)\Gamma z^*}\left\{\left(\frac{1}{\sigma} - \frac{1-u^*}{1-\eta}\right)\frac{\gamma\psi\eta}{u^*} + \frac{\eta(1-\gamma)+(\beta-\gamma\psi)}{1-\eta}\right\}. \tag{15}$$

The terms in the large braces in $DetJ$ are positive under Condition S. Therefore, like the sign of BJ, $DetJ >$ (*resp.* $<$)0 if $\Gamma <$ (*resp.* $>$)0.

We first rule out the case with three negative roots (the first row in Table 1). This case arises under $DetJ<0$, which requires condition $\Gamma>0$, and together (14) indicates $BJ<0$. However, the existence of three negative roots requires $TrJ<0$ and $-BJ+DetJ/TrJ<0$, that contradicts with $DetJ<0$ and $BJ<0$.

Next, suppose that the BGP is not a source. Then, the equilibrium path toward BGP is either determinate or indeterminate. Table 1 indicates that the equilibrium path toward BGP is determinate if $DetJ<0$ and indeterminate if $DetJ>0$. As a result, local indeterminacy arises when $\Gamma<0$. Therefore, we obtain

Proposition 3. *Under Assumption 1 and Condition S, local indeterminacy arises under* $\Gamma<0$.

We now investigate under what conditions $\Gamma<0$. Substituting Λ in (9a) into (9b) yields

$$\Gamma \equiv \{\Delta_s \Delta_p v^*\} + \left\{-\frac{\gamma\psi}{1-\gamma}\alpha(1-\eta)\right\}, \tag{16}$$

where $\Delta_p \equiv \alpha(1-\eta) - \eta\beta$ and

$$\Delta_s \equiv \left(\alpha + \frac{\alpha\gamma}{1-\gamma}\right)(1-\eta) - \eta\left(\beta + \frac{\beta\gamma}{1-\gamma} - \frac{\gamma\psi}{1-\gamma}\right) = \frac{1}{1-\gamma}[\alpha(1-\eta) - \eta(\beta - \gamma\psi)].$$

If $\Delta_p > (resp. <)0$, Sector Y is more physical (human) capital intensive from the *private* perspective. If $\Delta_s > (resp. <)0$, then Sector Y is more physical (human) capital intensive from the *social* perspective.[9]

If there is no productive public goods, then $\gamma = 0$ and (16) only involves the terms in the first large braces. The model is thus reduced to an otherwise standard model with $\Gamma = {\Delta_p}^2 v > 0$ (e.g., Rebelo, 1991). It follows that the equilibrium path toward BGP is determinate. Alternatively, if the public services are productive but not congestive, then $\gamma > 0$ and $\psi = 0$. Thus, (16) still involves only the terms in the first large braces with $\Gamma = \Gamma' \equiv {\Delta_p}^2 v/(1-\gamma) > 0$, so $DetJ < 0$. As a result, the equilibrium path toward BGP remains determinate.

We must point out that when $\gamma > 0$ and $\psi = 0$, the production function is affected by productive public good in the fashion of sector-specific externalities (cf, (6)). According to conventional wisdom (e.g., Benhabib et al., 2000; Mino, 2001), sector-specific externalities may change the factor intensity ranking in the social perspective different from that in the private perspective (i.e. $\Delta_p < 0$ and $\Delta_s > 0$), and thereby establish local indeterminacy. However, sector-specific externalities generated by productive public good do not work in that way in our model. The reason is that productive public services only contributes small sector-specific externalities via sector physical and human capital as long as $0 < \gamma < 1$, so that the effect cannot make the sign of Δ_s to deviate from the sign of Δ_p. As a result, local indeterminacy cannot emerge in a two-sector, human capital enhanced growth model with sector-specific externalities arising only from productive public good.

We now consider congestible public good services, and thus $\gamma > 0$ and $\psi > 0$. Then, (16) involves terms in both the first and second braces, with a smaller value in the first braces and a negative value in the second braces. If the effect in the second braces dominates that in the first braces, Γ is negative, and thus $DetJ$ is positive. As a result, the equilibrium path toward BGP is indeterminate.

Examining (16), the direct negative effects of γ andψ on Γ are[10]:

$$\left.\frac{\partial \Gamma}{\partial \gamma}\right|_{v=\bar{v}^*} = \frac{-1}{(1-\gamma)^2}\left\{\eta^2(1-\gamma)^2(1-\psi)\bar{v}^* + \alpha(\psi[1-\eta(1-v)] - \alpha\bar{v}^*)\right\} < 0 \text{ if } \psi > \underline{\psi}, \tag{17a}$$

[9]Dividing (7b) by (7c) yields$\frac{K_Y/H_Y}{K_X/H_X} = \frac{vK/(uH)}{(1-v)K/[(1-u)H]} = \frac{v(1-u)}{u(1-v)} = \frac{\alpha(1-\eta)}{\eta\beta}$. If $\Delta_p > 0$, then $\frac{K_Y/H_Y}{K_X/H_X} > 1$ and hence Sector Y is more physical capital intensive from the private perspective. Alternatively, from the social perspective, $\frac{K_Y^S/H_Y^S}{K_X^S/H_X^S} = \frac{v^S K^S/(u^S H^S)}{(1-v^S)K^S/[(1-u^S)H^S]} = \frac{v^S(1-u^S)}{u^S(1-v^S)} = \frac{\alpha(1-\eta)}{\eta(\beta-\gamma\psi)}$. If $\Delta_s > 0$, then $\frac{K_Y^S/H_Y^S}{K_X^S/H_X^S} > 1$ and hence Sector Y is more physical capital intensive from the social perspective.

[10]Both γ and ψ also have an indirect positive or negative effect on Γ via their effect on v^*, but the effect is associated with $\Delta_p\Delta_s$ that is very small.

$$\left.\frac{\partial \Gamma}{\partial \psi}\right|_{v=\bar{v}^*} = \frac{-\gamma}{(1-\gamma)^2}\left\{\eta[\alpha - \eta(1-\gamma)]\bar{v}^* + \alpha(1-\eta)\psi\right\} < 0, \text{ if } \psi > \underline{\psi}. \tag{17b}$$

where

$$\underline{\psi} \equiv \max\left\{\frac{\alpha\bar{v}^*}{1-\eta(1-v)}, \frac{\eta[\eta(1-\gamma)-\alpha]\bar{v}^*}{\alpha(1-\eta)}\right\}.$$

The results indicate that if the congestion effect (ψ) is large enough, both a higher degree of public spending effect (γ) and a higher degree of congestion effect (ψ) increase the possibility of $\Gamma < 0$, thereby implying a trade-off relationship between γ and ψ in order for $\Gamma < 0$. Moreover, the factor intensity rankings between sectors are not required for $\Gamma < 0$ as regardless of the signs of Δ_p and Δ_s, Γ is negative when γ and ψ are large enough.

Is it possible to simplify our model by employing only the negative aggregate externality without congestible public goods while maintaining technologies with social constant returns? It is not possible unless we allow for the goods production functions to be of increasing returns from the private perspectives, an improper setup for a competitive firm.[11] With productive public goods, there are positive sector-specific externalities so that our model avoids a production function of private increasing returns. As a result, congestible productive public goods are a mechanism to establish local indeterminacy. Thus, we obtain

Proposition 4. *Under Assumption 1 and Condition S, in a two-sector, human capital enhanced growth model with congestible productive public goods, the equilibrium path is locally indeterminate if the degrees of public spending externalities and congestion effects are large enough, regardless of relative factor intensity rankings.*

3.2. The intuition

The possibility of locally indeterminate equilibrium paths emerges in our model when the marginal product from the firm's perspective deviates from that from the social perspective. Denote as MPK_i the marginal product of physical capital in Sector $i = Y$ and X.[12] The relative marginal product of physical capital between Sectors Y and X is lower when more physical capital is reallocated from Sector X toward Sector Y, and thus MPK_y/MPK_x is decreasing in $vK/[(1-v)K]$, according to the law of diminishing marginal returns. However, the law of diminishing return fails to hold from the social perspective if the degrees of public good externality and congestion effects are large enough.

To see the point, (7b) or (9a) lead to

$$\frac{MPK_y}{MPK_x} = \Phi_1 u^{\psi\gamma/(1-\gamma)}\left(\frac{vK}{uH}\right)^{\Lambda}. \tag{18}$$

[11] If there is only a congestion effect without productive public services (i.e., $\gamma = 0$ and $\psi > 0$), then the technology with *social* constant returns requires $\alpha + \beta - \psi = 1$, which thus implies $\alpha + \beta > 1$; i.e. a technology with *private* increasing returns.

[12] Alternatively, we may use the relative marginal product of human capital to illustrate the intuition.

Differentiating (18) yields

$$\frac{\partial(MPK_y/MPK_x)}{\partial(vK/[(1-v)K])} = (1-v)^2\frac{\partial(MPK_y/MPK_x)}{\partial v} = \frac{-(1-u)(1-v)\Gamma}{\alpha(1-\eta)v} \gtreqless 0 \text{ if } \Gamma \lesseqgtr 0, \tag{19}$$

that indicates that if $\Gamma<0$, the relative marginal product of physical capital is increasing in $vK/[(1-v)K]$.

The reason for the above result is that, under the requirement of constant returns to scale in a technology, the public service congestion reduces the contribution of public capital services and thus increases the contribution of physical and human capital from the social perspective, thereby mitigating the speed of diminishing marginal product. When the degrees of public service externalities and congestion effects are sufficiently large, the effect of diminishing return from the private perspective is dominated. Similarly, with Pareto complements in physical and human capital in the technology, the relative marginal product of human capital is also increasing in $uH/[(1-u)H]$ from the social perspective if the degrees of public service externalities and congestion effects are sufficiently large.

Therefore, when the representative agent expects higher relative returns in Sector Y, it will allocate more resources from Sector X to Sector Y if the degrees of public good externality and congestion effects are large enough so that the law of diminishing return does not hold from the social perspectives. Then, higher vK and uH in Sector Y make the marginal returns of physical capital and human capital higher in Sector Y. As a consequence, expectations are self-fulfilling in equilibrium.

To quantitatively assess whether the degrees of both public spending externality and congestion are in a plausible range in order to establish local indeterminacy, we calibrate the model economy and quantitatively envisage the model in the next section.

4. Numerical simulations

4.1. Calibration

We now calibrate the model economy. We start by choosing parameter values, followed by solving the endogenous variables in a BGP. Finally, we establish the values of TrJ, BJ and $DetJ$ of the Jacobean matrix J, and determine the signs of eigenvalues.

We calibrate the model based upon the following parameter values representative of the economy in the US and consistent with a 2% long-run, real economic growth rate (Table 2).

The total tax revenues in the US, on average, account for 20% of its GDP after 1980, and hence $\tau = 20\%$ is chosen. Public investment is 1.7% of GDP after 1980 according to Tanzi and Schuknecht (2000) and hence the fraction of public spending invested in Sector Y is set as $\phi = 0.085$. Following Turnovsky (2000) we choose the degree of the externality of public spending upon the goods sector at $\gamma = 0.08$.

Table 2
Benchmark parameter values

Growth rate	τ	ϕ	γ	ψ	ρ	σ	α	η	A
0.02	0.2	0.085	0.08	0.71	0.025	1.5	0.3	0.2	0.05530

Finally, McMillan (1989) estimated the degree of congestion in fire protection services at the range of 0.499 and 0. 921. We choose $\psi = 0.71$, the mean value, for the degree of congestion. We must point out that the resulting calibrated value for A is insensitive to a different value of ϕ and ψ.

For the time preference rate, we set $\rho = 0.025$ in accordance with Benhabib and Perli (1994). As the intertemporal elasticity of substitution is usually smaller than one, we choose $\sigma = 1.5$, a value consistence with Jones et al. (1993) and Ben-Gad (2003). Finally, following Ben-Gad (2003), Benhabib and Perli (1994), and Raurich (2001), we set $\alpha = 0.3$ and $\eta = 0.2$.[13] With the degree of public spending externality and under a constant return to scale technology, the share of human capital in the goods sector, β, is implied. Using the above parameter values, we calibrate the productivity coefficient in the educational sector at $A = .0553$. With the parameter values, we then determine the endogenous variables in a BGP.

4.2. Local indeterminacy

We now simulate the model to see whether the BGP is a source, or the equilibrium path toward BGP is determinate or indeterminate. In the simulation, we maintain all the parameter values in Table 2 except for parameter values concerning congestible public good, γ and ψ. In Fig. 1, we simulate the region in the (γ, ψ) plane separating equilibrium paths in the neighborhood of the unique BGP that is a sink and a saddle, with the shaded area exhibiting local indeterminacy. Three observations are in order.

First, the equilibrium path to the unique BGP is indeterminate under plausible values for the degree of public spending externality and the degree of congestion. If the degree of public spending externality is high (say $\gamma = 30\%$), for local indeterminacy the required degree of congestion is as low as $\psi = 28.4\%$. Alternatively, if the congestion is a direct proportion to aggregate activities (i.e. $\psi = 1$), like the setup of Barro and Sala-i-Martin (1992) and Glomm and Ravikumar (1994), for local indeterminacy the required degree of public spending externality is as low as $\gamma = 3.6\%$.

Second, there is a trade-off between the degree of public service externality and the degree of congestion. For the required degree of public service externality at the

[13] While Ben-Gad (2003) chose $\alpha = 0.285$ and $\eta = 0.2$, Raurich (2001) chose $\alpha = 0.42$ and $\eta = 0.1$ in his decentralized economy and $\alpha = 0.35$ and $\eta = 0.2$ in his centralized economy. Benhabib and Perli (1994) used $\alpha = 0.25$. Thus, our value of α and η lies in the range of the values in these works. Moreover, we will change the value of α and η to assure the robustness of our quantitative results.

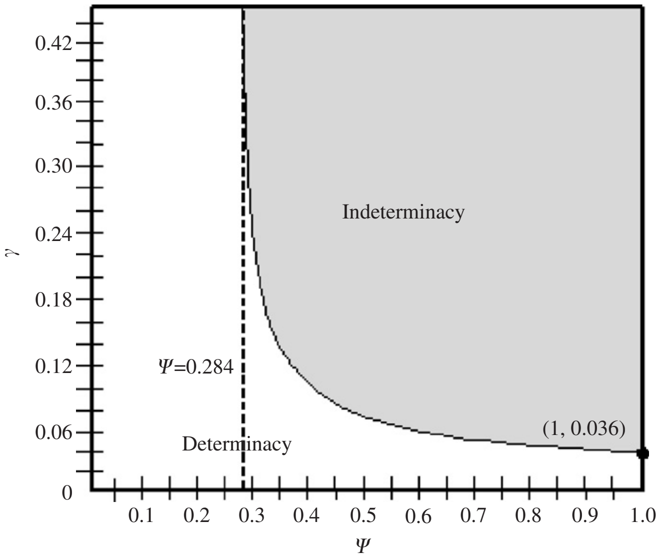

Fig. 1. *Note*: (1) $A = 0.0553$, $\tau = 0.2$, $\phi = 0.085$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$. (2) γ represents the degree of externality of productive public spending and ψ represents the degree of congestion. (3) The shaded area presents the region of indeterminacy and $\psi = 0.284$ is the minimum of ψ.

value of $\gamma \in [0.1, 0.15]$, estimated by Shioji (2001) using a panel data of US and Japan, and of $\gamma = 0.2$, by Lynde and Richmond (1993b) using time-series data for the US, the required degree of congestion for local indeterminacy is about $\psi = 35\%$, a value lower than the estimate of $\psi \in [0.499, 0.921]$ by McMillan (1989) using the data of fire protection services. The parameter values of γ and ψ are plausible and reasonable.

Finally, there is no possibility of a source in the simulation.

Moreover, from the aspect of an aggregate production (cf. (6)), the externalities include two sources of positive sector-specific externalities ($\alpha\gamma/(1-\gamma)$ and $\beta\gamma/(1-\gamma)$) and one source of negative aggregate externality, $\psi\gamma/(1-\gamma)$. Fig. 2 illustrates the degrees of the three externalities necessary for local indeterminacy in the bold segment of the lines. The required degrees of the externalities for indeterminacy are as low as 1.1% for the sector-specific externality associated with sector physical capital, 2.6% for the sector-specific externality associated with sector human capital, and 3.7% for the negative externality associated with the aggregative activities.

Finally, in order to assure the robustness of the above quantitative results, we conduct some sensitivity analysis (Table 3). In the analysis, we change the value of one parameter from the benchmark case while keeping the values of all other parameters unchanged, and calibrate the value for A in consistence with the 2% economic growth rate. We have experimented using a higher and lower time preference rate, a 50% increase and a 50% decrease in the reciprocal of the intertemporal elasticity of substitution, an increase and a decrease in the share of physical capital in Sectors Y and X. In all of the cases the results for indeterminacy

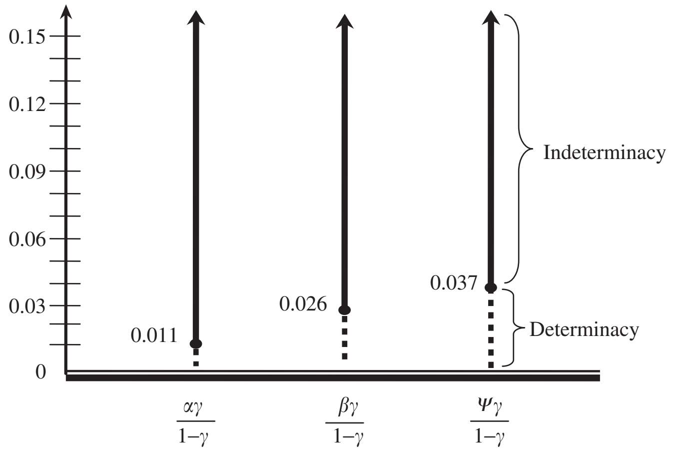

Fig. 2. *Note*: (1) $A = 0.0553$, $\tau = 0.2$, $\phi = 0.085$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$. (2) Values $\alpha\gamma/(1-\gamma)$ and $\beta\gamma/(1-\gamma)$ are the degree of sector-specific externalities associated with vK and uH, respectively, and $\psi\gamma/(1-\gamma)$ is the degree of negative aggregate externality. (3) The bold segment is for indeterminacy, and 0.011, 0.026 and 0.037, respectively, are the minimal required value of each kind of externalities to establish local indeterminacy.

Table 3
Robustness analysis

Variations	A	Lower Bound of ψ	Minimum of γ	Minimum of sector specific externalities $\left(\frac{\alpha\gamma}{1-\gamma}, \frac{\beta\gamma}{1-\gamma}\right)$	Minimum of negative aggregate externality $\frac{\psi\gamma}{1-\gamma}$	Region for indeterminacy
Benchmark[a]	0.055	0.28	0.036	(0.011, 0.026)	0.037	Figs. 1 and 2
$\rho = 0.01$	0.036	0.27	0.032	(0.009, 0.023)	0.033	Similar
$\rho = 0.05$	0.089	0.31	0.039	(0.012, 0.028)	0.040	Similar
$\sigma = 1.1$	0.045	0.30	0.034	(0.010, 0.024)	0.034	Similar
$\sigma = 3$	0.097	0.29	0.040	(0.012, 0.029)	0.041	Similar
$\alpha = 0.25(>\eta)$	0.056	0.16	0.010	(0.002, 0.007)	0.009	Similar
$\alpha = 0.42(>\eta)$	0.050	0.57	0.145	(0.071, 0.098)	0.169	Similar
$\eta = 0.1(<\alpha)$	0.059	0.48	0.142	(0.049, 0.115)	0.165	Similar
$\eta = 0.25(<\alpha)$	0.051	0.18	0.010	(0.003, 0.007)	0.009	Similar
$\eta = 0.3(=\alpha)$	0.048	0.10	0.010	(0.003, 0.006)	0.001	Similar
$\eta = 0.35(>\alpha)$	0.044	0.10	0.010	(0.003, 0.007)	0.004	Similar
$\eta = 0.4(>\alpha)$	0.040	0.10	0.033	(0.010, 0.023)	0.015	Similar[b]

[a]Benchmark parameters: $\tau = 0.2$, $\phi = 0.085$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$.
[b]Source emerges when γ is large enough and ψ is small enough in this case.

are quantitatively similar to those in Figs. 1 and 2. We illustrate in Table 3 the resulting lower bounds of sector specific externalities and negative aggregate externalities. Two remarks are in order.

First, the minimal required degrees of externalities are lowered as the difference between α and η decreases. In particular, under $\alpha = \eta = 0.3$, the sector-specific externalities associated with physical and human capital are as low as 0.3% and 0.6%, respectively, and the negative externality associated with aggregate activities is reduced to be 0.1%.

Second, local indeterminacy arises regardless of the factor intensity rankings. The otherwise required condition for local indeterminacy in existing literature that Sector Y be human-capital-intensive from private perspectives ($\alpha(1-\eta)<\eta\beta$), is now not necessary.

5. Concluding remarks

In this paper, we construct a two-sector, constant-return growth model with congestible productive public goods. While the productive public good exerts positive sector-specific externalities, the congestion effect generates negative aggregate externalities. The sector-specific externalities alone arising from productive public goods can not establish local indeterminacy without the combination of negative externality, unless we allow for a technology with increasing returns from the private perspectives, an improper setup for a competitive firm. We find that if the degrees of public good externality and congestion effects are sufficiently large, congestible public good that combines both positive and negative externalities is a mechanism that can establish local indeterminacy.

We have calibrated the model based on the US economy and simulate the values of the degrees of public goods services and congestion effects. The required degrees of public good externality and congestion effects are consistent with the estimated values in existing empirical studies, and are thus plausible.

There are limitations in the present model. In our setup, the public spending is useful only in the final goods sector. In a working paper version of this paper (Chen and Lee, 2005), we have assumed that public goods are useful to both the final goods and the education sectors. The public goods there are subject to congestion effects associated with aggregate activities that are represented by both aggregate physical and human capital, like that in Appendix C. The results are the same, but the algebra and the analysis are complicated. The present version simplifies the economic structure a great deal while maintaining the key ingredients.

Finally, we implicitly assume that there is no market for education and thus no income taxation is made on the education sector. The setup is different from Bond et al. (1996) where all factor incomes from both sectors are taxed. If we relax our present assumption and allow for the government to tax both outputs while maintaining other setup unchanged, the conditions for local indeterminacy is more stringent: it requires too high a degree of congestion. See Appendix E. Alternatively, we may also change for formulation for the congestion effects in (2) and allow for not only aggregate congestion but also relative congestion.[14] The conditions for local

[14] Fisher and Turnovsky (1998) and Eicher and Turnovsky (2000) have introduced relative congestion into their model to analyze the effect of fiscal policy on growth rate.

indeterminacy are met if there are sufficiently large values for the degree of either absolute or relative congestion. In particular, for a given degree of relative congestion effects, we find that all the properties for local indeterminacy in the paper remain to hold true. See Appendix F.

Appendix A

This appendix derives how the transformed system (11a)–(11c) is obtained.

First, when we substitute (7b) into (7d), and (7c) into (7e), together (1a,b) and (5), we obtain:

$$(1-\tau)\alpha\frac{y}{vk} = (1-\tau)\alpha(\phi\tau)^{\gamma/(1-\gamma)}u^{\gamma\psi/(1-\gamma)}\left(\frac{vk}{uh}\right)^{-(\beta-\gamma\psi)/(1-\gamma)} \equiv (1-\tau)r = \rho - \frac{\dot{\lambda}}{\lambda}, \tag{A.1}$$

$$(1-\eta)\frac{x}{(1-u)h} = (1-\eta)A\left(\frac{\eta\beta}{\alpha(1-\eta)}\right)^{\eta}\left(\frac{vk}{uh}\right)^{\eta} \equiv w = \rho - \frac{\dot{\mu}}{\mu}, \tag{A.2}$$

Next, we use (A.1) and (3a), and (A.2) and (3b) to derive

$$\frac{\dot{K}}{K} = \frac{v(1-\tau)r}{\alpha} - \frac{s}{z}, \tag{A.3}$$

$$\frac{\dot{H}}{H} = \frac{(1-u)w}{\theta}, \tag{A.4}$$

where r and w are as defined in (10) and (11a), with $s \equiv C/H$ and $z \equiv K/H$.

Third, with the help of (A.1) and (A.2), we rearrange (7b) to obtain

$$\frac{\mu}{\lambda} \equiv p = \Phi_1 u^{\psi\gamma/1-\gamma}\left(\frac{vk}{uh}\right)^{\Lambda}, \tag{A.5}$$

where Φ_1 and Λ are as defined in (9a). (A.5) indicates that vk/uh is a function of u and p, and is easily rewritten as the form in (9a).

Finally, the system (11a–c) is derived as follows. We derive (11a) using (A.1), (A.2) and (A.5), (11b) using (10) and (A.4), and finally (11c) using (A.3) and (A.4).

Appendix B

The model when the congestion effect is represented by aggregate physical capital.

The perceived amount of public goods services received by the representative agent becomes

$$G_y = \phi G\left(\frac{1}{K}\right)^{\psi}, \tag{B.1}$$

where K is the aggregate physical capital. All other setups are the same as those in the main text.

Together (B1) and (5) the production technology in Sector Y is

$$y = (\phi\tau)^{\gamma/(1-\gamma)}(vk)^{\alpha}(uh)^{\beta}(vK)^{\alpha\gamma/(1-\gamma)}(uH)^{\beta\gamma/(1-\gamma)}(K)^{-\psi\gamma/(1-\gamma)}. \tag{B.2}$$

The first-order conditions for the representative agent's problem are the same as (7).

After the transformation, the equilibrium system is as follows:

$$\frac{\dot{p}}{p} = \frac{\dot{\mu}}{\mu} - \frac{\dot{\lambda}}{\lambda} = (1-\tau)r(u(p,z),p) - w(u(p,z),p), \tag{B.3}$$

$$\frac{\dot{s}}{s} = \frac{\dot{C}}{C} - \frac{\dot{H}}{H} = \frac{(1-\tau)r(u(p,z),p) - \rho}{\sigma} - \frac{1-u(p,z)}{1-\eta}w(u(p,z),p), \tag{B.4}$$

$$\frac{\dot{z}}{z} = \frac{\dot{K}}{K} - \frac{\dot{H}}{H} = \frac{v(u(p,z))}{\alpha}(1-\tau)r(u(p,z),p) - \frac{s}{z} - \frac{1-u(p,z)}{1-\eta}w(u(p,z),p), \tag{B.5}$$

where

$$r(u(p,z),p) = \Phi_2 p^{-(\beta-\gamma\psi)/\Lambda(1-\gamma)}\{v[u(p,z)]\}^{\gamma\psi(1-\eta)/\Lambda(1-\gamma)} \gtrless 0,$$

$$w(u(p,z),p) = \Phi_3 p^{\eta/\Lambda}\{v[u(p,z)]\}^{-\gamma\psi\eta/\Lambda(1-\gamma)} \gtrless 0.$$

We have used relationship

$$\Phi_4\left(\frac{1}{\sigma} - \frac{1-u}{1-\eta}\right)v(u)^{\gamma\psi\eta/\eta(1-\gamma)+\beta-\gamma\psi} = \frac{\rho}{\sigma}$$

to prove the existence and uniqueness of a BGP. The local dynamics of the economical system are characterized by

$$\begin{pmatrix} \dot{p} \\ \dot{s} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} J_{11} & 0 & J_{13} \\ J_{21} & 0 & J_{23} \\ J_{31} & -1 & J_{33} \end{pmatrix}\begin{pmatrix} p-p^* \\ s-s^* \\ z-z^* \end{pmatrix}, \tag{B.6}$$

where

$$J_{11} = \frac{-w^*}{(1-\gamma)\Lambda\Gamma}\left\{\gamma\psi\alpha(1-\eta)\frac{1-v}{1-u} + \Gamma[\eta(1-\gamma)+\beta-\gamma\psi]\right\},$$

$$J_{13} = \frac{\alpha(1-\eta)\gamma\psi w^* p^*}{(1-\gamma)\Gamma z^*}\frac{1-v}{1-u} \gtrless 0, \text{ if } \Gamma \gtrless 0,$$

$$J_{21} = \frac{-s^* w^*}{\Gamma p^*}\left\{\frac{\gamma\psi\alpha(1-\eta)}{(1-\gamma)\Lambda}\left[\frac{1-\eta}{\sigma} + \frac{\eta(1-u^*)}{1-\eta}\right]\frac{1-v}{1-u} + \frac{\Gamma}{(1-\gamma)\Lambda}\left[\frac{\beta-\gamma\psi}{\sigma} + \frac{\eta(1-\gamma)(1-u^*)}{1-\eta}\right] + \alpha u^*\right\},$$

$$J_{23} = \frac{\alpha s^* w^*}{\Gamma z^*}\left\{\frac{(1-\eta)\gamma\psi}{1-\gamma}\left[\frac{1-\eta}{\sigma}+\frac{\eta(1-u^*)}{1-\eta}\right]\frac{1-v}{1-u}+\Lambda u^*\right\},$$

$$J_{31} = \frac{-z^* w^*}{\Gamma p^*}\left\{\frac{\gamma\psi\alpha(1-\eta)}{(1-\gamma)\Lambda}\left[\frac{(1-\eta)v^*}{\alpha}+\frac{\eta(1-u^*)}{1-\eta}\right]\frac{1-v}{1-u}\right.$$
$$\left.\times\left[\frac{(\beta-\gamma\psi)v^*}{\alpha}+\frac{\eta(1-\gamma)(1-u^*)}{1-\eta}\right]+\alpha u^*+\frac{(1-\eta)v^*(1-v^*)}{1-u^*}\right\},$$

$$J_{33} = \frac{w^*}{\Gamma}\left\{\alpha(1-\eta)\left[\frac{\gamma\psi}{1\text{-}\gamma}\left(\frac{(1-\eta)v^*}{\alpha}+\frac{\eta(1-u^*)}{1-\eta}\right)\frac{1-v}{1-u}\right.\right.$$
$$\left.\left.+\Lambda\left(\frac{v^*(1-v^*)}{\alpha(1-u^*)}+\frac{u^*}{1-\eta}\right)\right]+\frac{\Gamma s^*}{z^*}\right\},$$

$$\Lambda = \frac{(\alpha-\gamma\psi)(1-\eta)-\eta\beta}{1-\gamma} = \left(\alpha+\frac{\alpha\gamma}{1-\gamma}-\frac{\gamma\psi}{1-\gamma}\right)(1-\eta)-\eta\left(\beta+\frac{\beta\gamma}{1-\gamma}\right),$$

$$\Gamma = \Lambda[\alpha(1-\eta)-\eta\beta]v-\frac{\gamma\psi}{1-\gamma}\alpha(1-\eta)\frac{1-v}{1-u}.$$

The determinant of the bordered Hessian of the Jacobean matrix is

$$BJ = \frac{-s^* w^*}{(1-\gamma)\Gamma z^*}\left\{\alpha\gamma\psi\frac{1-v}{1-u}\left[(1-\eta)\left(1-\frac{1-\eta}{\sigma}\right)+\eta(u^*-\eta)\right]\right.$$
$$\left.+\frac{[\eta(1-\gamma)+(\beta-\gamma\psi)]w^* z^*}{\alpha s^* u^*}M(u^*)\right\}. \tag{B.7}$$

It follows that $BJ>$(*resp.* $<$)0 if $\Gamma<$ (*resp.* $>$) 0. Moreover, the determinant of the Jacobean is

$$DetJ = \frac{-\alpha(1-\eta)u^* s^* w^{*2}}{(1-\gamma)\Gamma z^*}\left\{\left(\frac{1}{\sigma}-\frac{1-u^*}{1-\eta}\right)\frac{\gamma\psi\eta}{u^*}\frac{1-v}{1-u}+\frac{\eta(1-\gamma)+(\beta-\gamma\psi)}{1-\eta}\right\}. \tag{B.8}$$

The terms in the large braces in $DetJ$ are positive under Condition S. Therefore, like the sign of BJ, $DetJ>$ (*resp.* $<$)0 if $\Gamma<$ (*resp.* $>$)0. Thus, the results in Proposition 3 hold in this case.

We now investigate under what conditions $\Gamma<0$. Rewriting Γ yields

$$\Gamma \equiv \{\Delta_s\Delta_p v^*\}+\left\{-\frac{\gamma\psi}{1-\gamma}\alpha(1-\eta)\frac{1-v}{1-u}\right\}, \tag{B.9}$$

where $\Delta_p \equiv \alpha(1-\eta)-\eta\beta$ and

$$\Delta_s \equiv \left(\alpha+\frac{\alpha\gamma}{1-\gamma}-\frac{\gamma\psi}{1-\gamma}\right)(1-\eta)-\eta\left(\beta+\frac{\beta\gamma}{1-\gamma}\right) = \frac{(\alpha-\gamma\psi)(1-\eta)-\eta\beta}{1-\gamma}.$$

The fundamental structure of Γ is identical to that in (16). Therefore, the results in Proposition 4 maintain hold true and the outcome of numerical simulation is similar.

Appendix C

The model when the congestion effect is represented by aggregate physical and human capital.

The perceived amount of public goods services received by the representative agent is

$$G_y = \phi G\left(\frac{1}{KH}\right)^{\psi}, \tag{C.1}$$

where K is the aggregate physical capital. Other basic setups are the same.

Together (C1) and (5) the production technology in Sector Y is

$$y = (\phi\tau)^{\gamma/(1-\gamma)}(vk)^{\alpha}(uh)^{\beta}(vK)^{\alpha\gamma/(1-\gamma)}(uH)^{\beta\gamma/(1-\gamma)}(KH)^{-\psi\gamma/(1-\gamma)}. \tag{C.2}$$

We have shown that relationship

$$\Phi_4\left(\frac{1}{\sigma} - \frac{1-u}{1-\eta}\right)[v(u)u]^{\gamma\psi\eta/[\eta(1-\gamma)+\beta-\gamma\psi]} = \frac{\rho}{\sigma},$$

has a unique BGP.

For the local dynamics of the economical system, we have shown $BJ >$ (*resp.* $<$)0 and $DetJ >$ (*resp.* $<$)0 if $\Gamma <$ (*resp.* $>$) 0. Thus, the claim in Proposition 3 holds. We may rewrite Γ as

$$\Gamma \equiv \{\Delta_s \Delta_p v^*\} + \left\{-\frac{\gamma\psi}{1-\gamma}\alpha(1-\eta)\left(1+\frac{1-v}{1-u}\right)\right\}, \tag{C.3}$$

where $\Delta_p \equiv \alpha(1-\eta) - \eta\beta$ and

$$\Delta_s \equiv \left(\alpha + \frac{\alpha\gamma}{1-\gamma} - \frac{\gamma\psi}{1-\gamma}\right)(1-\eta) - \eta\left(\beta + \frac{\beta\gamma}{1-\gamma} - \frac{\gamma\psi}{1-\gamma}\right) = \frac{(\alpha-\gamma\psi)(1-\eta) - \eta(\beta-\gamma\psi)}{1-\gamma}.$$

The fundamental structure of Γ is identical to that in (16). Therefore, the claims in Proposition 4 hold, and the outcome of numerical simulation is similar.

Appendix D

The model with the depreciation for physical and human capital.

The basic setups are the same, except for the laws of motions in (3a) and (3b) modified as

$$\dot{k} = (1-\tau)y - c - \delta_k k, \tag{D.1}$$

$$\dot{h} = x - \delta_h h, \tag{D.2}$$

where δ_k and δ_h are the depreciation rate of physical and human capital, respectively.

The necessary conditions for the representative agent's problem are (7a)–(7c) and

$$\lambda\left[(1-\tau)\alpha\frac{y}{k} - \delta_k\right] + \mu\eta\frac{x}{k} = \rho\lambda - \dot{\lambda}, \tag{D.3}$$

$$\lambda(1-\tau)\beta\frac{y}{h}+\mu\left[(1-\eta)\frac{x}{h}-\delta_h\right]=\rho\mu-\dot{\mu}. \tag{D.4}$$

First, when we substitute (7b) into (D3), and (7c) into (D4), with (1a,b) and (5), we obtain

$$(1-\tau)\alpha\frac{y}{vk}=(1-\tau)\alpha(\phi\tau)^{\gamma/(1-\gamma)}u^{\gamma\psi/(1-\gamma)}\left(\frac{vk}{uh}\right)^{-(\beta-\gamma\psi)/(1-\gamma)}\equiv(1-\tau)r=\rho+\delta_k-\frac{\dot{\lambda}}{\lambda}. \tag{D.5}$$

$$(1-\eta)\frac{x}{(1-u)h}=(1-\eta)A\left(\frac{\eta\beta}{\alpha(1-\eta)}\right)^{\eta}\left(\frac{vk}{uh}\right)^{\eta}\equiv w=\rho+\delta_h-\frac{\dot{\mu}}{\mu}, \tag{D.6}$$

Next, (7a) and (D5) lead to the following relationship:

$$\frac{\dot{C}}{C}=\frac{1}{\sigma}[(1-\tau)r(u(p,z),p)-\rho-\delta_k]. \tag{D.7}$$

Finally, we use (D5) and (D1), and (D6) and (D1) to derive

$$\frac{\dot{K}}{K}=\frac{v(1-\tau)r}{\alpha}-\frac{s}{z}-\delta_k, \tag{D.8}$$

$$\frac{\dot{H}}{H}=\frac{(1-u)w}{\theta}-\delta_h, \tag{D.9}$$

Now we transform the equilibrium conditions into a stationary system.

Case D1. We start with the case when both depreciation rates are equal i.e. $\delta_{\mathrm{k}}=\delta_h=\delta$. The equilibrium system is the same as (11a)–(11c) except for (11b) modified as follows:

$$\frac{\dot{s}}{s}=\frac{\dot{C}}{C}-\frac{\dot{H}}{H}=\frac{(1-\tau)r(u(p,z),p)-\rho-\delta}{\sigma}-\frac{1-u(p,z)}{1-\eta}w(u(p,z),p)+\delta. \tag{D.10}$$

We have shown that relationship

$$LHS(u)\equiv\Phi_4\left(\frac{1}{\sigma}-\frac{1-u}{1-\eta}\right)(u)^{\gamma\psi\eta/\eta(1-\gamma)+\beta-\gamma\psi}=\frac{\rho-(\sigma-1)\delta}{\sigma}$$

has a unique BGP (see Appendix Fig. D1) if the following conditions hold.

Condition S1. . $\sigma\geqslant1$ and $\Phi_4>\rho-(\sigma-1)\delta\geqslant0$.

Condition S2. $\sigma\geqslant1$ and $LHS(\tilde{u})<[\rho-(\sigma-1)\delta]/\sigma$.

The results of the linear Taylor expansion of system (11a), (D10) and (11c) are exactly the same as (13), except for the number of BGP. In the case when $\rho\geqslant(\sigma-1)\delta$, the BGP is unique and the dynamic analysis for local indeterminacy is the same as in the main text. In the case when $\rho<(\sigma-1)\delta$, there are two BGPs. Then, we will show that at least one of the two BGPs is locally indeterminate by examining (15). Thus, we have shown local indeterminacy emerges easily.

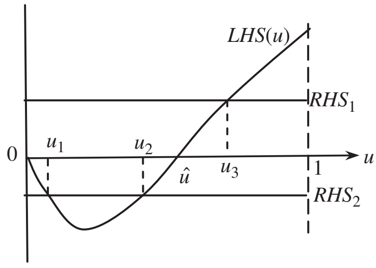

Fig. D1. $\delta_k = \delta_h$ and $\delta_k \neq \delta_h$.

Case D2. When the depreciation rates are different, i.e., $\Delta_d \equiv \delta_h - \delta_k \neq 0$, the stationary equilibrium system is

$$\frac{\dot{p}}{p} = \frac{\dot{\mu}}{\mu} - \frac{\dot{\lambda}}{\lambda} = (1-\tau)r(u(p,z),p) - w(u(p,z),p) + \Delta_d, \tag{D.11}$$

$$\frac{\dot{s}}{s} = \frac{\dot{C}}{C} - \frac{\dot{H}}{H} = \frac{(1-\tau)r(u(p,z),p) - \rho - \delta_k}{\sigma} - \frac{1-u(p,z)}{1-\eta} w(u(p,z),p) + \delta_h, \tag{D.12}$$

$$\frac{\dot{z}}{z} = \frac{\dot{K}}{K} - \frac{\dot{H}}{H} = \frac{v(u(p,z))}{\alpha}(1-\tau)r(u(p,z),p) - \frac{s}{z} - \frac{1-u(p,z)}{1-\eta} w(u(p,z),p) + \Delta_d. \tag{D.13}$$

Under a mild condition, we have shown there exist an interior BGP if $\rho \geqslant (\sigma-1)\delta$, and two interior BGPs if $\rho < (\sigma-1)\delta$, as illustrated in Appendix Fig. D1 where u is replaced by vk/uh.

To investigate the local dynamics of the economical system, we take a linear Taylor's expansion of (D11–13) in the neighborhood of a BGP. We have shown Condition S1 or S2 imply $BJ >$ (*resp.* $<$) 0 and $DetJ >$ (*resp.* $<$) 0 if $\Gamma <$ (*resp.* $>$) 0.

As the nature of BJ and $DetJ$ is similar to that in the main text, local dynamics is thus similar to that in the main text. We thereby calibrate the model following the parameter values used in Mulligan and Sala-i-Martin (1993) and Raurich (2001). We choose $\delta_k = 0.05$, and a slightly smaller depreciation rate of human capital at $\delta_h = 0.04$. Other parameter values are the same as the benchmark values in Table 3 in the main text. With the calibrated values of parameters, we then conduct the simulations with the results summarized in Appendix Table D1. We find it is easy to establish local indeterminacy.

Appendix E

The model when public spending is financed by income taxes in both sectors.

For simplicity, we assume the tax rates in the two sectors are identical, denoted as τ. The motions of the two kinds of capital stock for the representative

Table D1
Simulations

Variations	A	Lower Bound of ψ	Minimum of γ	Minimum of sector specific externalities $\left(\frac{\alpha\gamma}{1-\gamma}, \frac{\beta\gamma}{1-\gamma}\right)$	Minimum of negative aggregate externality $\frac{\psi\gamma}{1-\gamma}$	Region for indeterminacy
Benchmark[a]	0.117	0.23	0.027	(0.008, 0.019)	0.027	Similar to Figs. 1 and 2
$\rho = 0.01$	$\rho < (\sigma - 1)\delta_h$, indeterminacy arises definitely					
$\rho = 0.05$	0.157	0.24	0.028	(0.008, 0.020)	0.028	Similar
$\sigma = 1.1$	0.105	0.19	0.019	(0.005, 0.013)	0.018	Similar
$\sigma = 3$	$\rho < (\sigma - 1)\delta_h$, indeterminacy arises definitely					
$\alpha = 0.25(>\eta)$	0.118	0.12	0.010	(0.002, 0.007)	0.007	Similar
$\alpha = 0.42(>\eta)$	0.111	0.48	0.115	(0.054, 0.075)	0.129	Similar
$\eta = 0.1(<\alpha)$	0.113	0.40	0.114	(0.038, 0.090)	0.128	Similar
$\eta = 0.25(<\alpha)$	0.115	0.14	0.010	(0.003, 0.007)	0.006	Similar
$\eta = 0.3(=\alpha)$	0.112	0.08	0.010	(0.003, 0.006)	0.001	Similar
$\eta = 0.35(>\alpha)$	0.108	0.08	0.010	(0.003, 0.006)	0.004	Similar
$\eta = 0.4(>\alpha)$	0.103	0.09	0.025	(0.009, 0.013)	0.008	Similar
$\delta_k = 0.03(\delta_k < \delta_h)$	0.110	0.23	0.027	(0.008, 0.019)	0.027	Similar
$\delta_k = 0.1(\delta_k > \delta_h)$	0.131	0.23	0.027	(0.008, 0.019)	0.027	Similar

[a]Benchmark parameters: $\tau = 0.2$, $\phi = 0.085$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$, $\delta_k = 0.05$, $\delta_h = 0.04$.

agent become:

$$\dot{k} = (1-\tau)y - c, \tag{E.1}$$

$$\dot{h} = (1-\tau)x. \tag{E.2}$$

Then, the government budget constraint is

$$G = \tau(Y + pX), \tag{E.3}$$

where p which is the price of human capital relative to physical capital, which is equal to μ/λ.

All other setups are the same as those in the main text.

We have shown the existence and uniqueness of a BGP. We then investigate the local dynamics of the economical system by taking a linear Taylor's expansion of the transformed 3×3 system in the neighborhood of the unique BGP. We find under Condition S, $BJ >$ (*resp.* $<$)0 if $\Gamma <$ (*resp.* $>$) 0, where

$$\Gamma = \Delta[\alpha(1-\eta) - \eta\beta]v - \frac{\gamma}{1-\gamma}\alpha(1-\eta)\left[\psi - \frac{\beta}{(1-\eta)u + \beta(1-u)}\right].$$

We have also found the determinant of the Jacobean is similar to (15) in the main text, with the extra term

$$\left[\psi - \frac{\beta}{(1-\eta)u^* + \beta(1-u^*)}\right].$$

If this term is sufficiently positive, under Condition S, $DetJ>$ (*resp.* $<$)0 if $\Gamma<$ (*resp.* $>$)0. The claims in Propositions 3,4 remain true.

However, the condition

$$\psi > \frac{\beta}{(1-\eta)u^* + \beta(1-u^*)}$$

is more stringent to meet. Therefore, we modify the formulation of congestion in Appendix F as follows.

Appendix F

The model with income taxes in both sectors and both aggregate and relative congestion.

To consider both aggregate and relative congestion, we follow Eicher and Turnovsky (2000) and assume that the perceived amount of public goods services received by the representative agent becomes the following form:

$$G_y = \phi G\left(\frac{1}{H}\right)^{\psi}\left(\frac{uh}{H}\right)^{\theta}, \tag{F.1}$$

where $0<\psi<1$ and $\theta>0$ are respectively the degree of aggregate and relative congestion, and uh/H represents the relative congestion.

Using (F1), the production technology in Sector Y is thus

$$y = (\phi\tau)^{\gamma/(1-\gamma)}(vk)^{\alpha}(uh)^{\beta}(vK)^{\alpha\gamma/(1-\gamma)}(uH)^{\beta\gamma/(1-\gamma)}H^{-\psi\gamma/(1-\gamma)}\left(1+p\frac{X}{Y}\right)^{\gamma/(1-\gamma)}\left(\frac{uh}{H}\right)^{\gamma\theta/(1-\gamma)}. \tag{F.2}$$

The term

$$1+p\frac{X}{Y}$$

represents the intersectoral externality and uh/H represents the external effect of relative congestion. Other setups are the same as those in the main text.

We have derived the first-order conditions, and then transformed the equilibrium conditions into the following stationary system:

$$\frac{\dot{p}}{p} = \frac{\dot{\mu}}{\mu} - \frac{\dot{\lambda}}{\lambda} = (1-\tau)r(u(p,z),p) - (1-\tau)w(u(p,z),p), \tag{F.3}$$

$$\frac{\dot{s}}{s} = \frac{\dot{C}}{C} - \frac{\dot{H}}{H} = \frac{(1-\tau)r(u(p,z),p)-\rho}{\sigma} - \frac{1-u(p,z)}{1-\eta}(1-\tau)w(u(p,z),p), \tag{F.4}$$

$$\frac{\dot{z}}{z} = \frac{\dot{K}}{K} - \frac{\dot{H}}{H} = \frac{v(u(p,z))}{\alpha}(1-\tau)r(u(p,z),p) - \frac{s}{z} - \frac{1-u(p,z)}{1-\eta}(1-\tau)w(u(p,z),p), \tag{F.5}$$

where

$$r(u(p,z),p) = \Phi_2 p^{-(\beta-\gamma\psi)/\Lambda(1-\gamma)} u(p,z)^{\gamma(\psi+\theta)(1-\eta)/\Lambda(1-\gamma)} \times \left(1+\frac{\beta}{1-\eta}\frac{1-u}{u}\right)^{\gamma(1-\eta)/\Lambda(1-\gamma)} \geqslant 0,$$

$$w(u(p,z),p) = \Phi_3 p^{\eta/\Lambda} u(p,z)^{-\gamma\eta(\psi+\theta)/\Lambda(1-\gamma)} \left(1+\frac{\beta}{1-\eta}\frac{1-u}{u}\right)^{-\gamma\eta/\Lambda(1-\gamma)} \geqslant 0,$$

with

$$\Phi_3 = (1-\tau)(1-\eta)\left[\frac{\eta\beta}{\alpha(1-\eta)}\right]^{\eta} \Phi_1^{-\eta/\Lambda}.$$

We have shown the existence and uniqueness of a BGP. To investigate the local dynamics, we take a linear Taylor's expansion of (F3–5) in the neighborhood of the unique BGP to obtain

$$\begin{pmatrix} \dot{p} \\ \dot{s} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} J_{11} & 0 & J_{13} \\ J_{21} & 0 & J_{23} \\ J_{31} & -1 & J_{33} \end{pmatrix} \begin{pmatrix} p-p^* \\ s-s^* \\ z-z^* \end{pmatrix}, \tag{F.6}$$

where

$$J_{11} = \frac{-w^*}{(1-\gamma)\Lambda\Gamma}\left\{\gamma\alpha(1-\eta)\left[\psi+\theta-\frac{\beta}{(1-\eta)u^*+\beta(1-u^*)}\right] + \Gamma[\eta(1-\gamma)+\beta-\gamma\psi]\right\},$$

$$J_{13} = \frac{\alpha(1-\eta)\gamma w^* p^*}{(1-\gamma)\Gamma z^*}\left[\psi+\theta-\frac{\beta}{(1-\eta)u^*+\beta(1-u^*)}\right],$$

$$J_{21} = \frac{-s^* w^*}{\Gamma p^*}\left\{\frac{\gamma\alpha(1-\eta)}{(1-\gamma)\Lambda}\left[\frac{1-\eta}{\sigma}+\frac{\eta(1-u^*)}{1-\eta}\right]\left[\psi+\theta-\frac{\beta}{(1-\eta)u^*+\beta(1-u^*)}\right] + \frac{\Gamma}{(1-\gamma)\Lambda}\left[\frac{\beta-\gamma\psi}{\sigma}+\frac{\eta(1-\gamma)(1-u^*)}{1-\eta}\right] + \alpha u^*\right\},$$

$$J_{23} = \frac{\alpha s^* w^*}{\Gamma z^*}\left\{\frac{(1-\eta)\gamma}{1-\gamma}\left[\frac{1-\eta}{\sigma}+\frac{\eta(1-u^*)}{1-\eta}\right]\left[\psi+\theta-\frac{\beta}{(1-\eta)u^*+\beta(1-u^*)}\right] + \Lambda u^*\right\},$$

$$J_{31} = \frac{-z^* w^*}{\Gamma p^*}\left\{\frac{\gamma\alpha(1-\eta)}{(1-\gamma)\Lambda}\left[\frac{(1-\eta)v^*}{\alpha}+\frac{\eta(1-u^*)}{1-\eta}\right]\left[\psi+\theta-\frac{\beta}{(1-\eta)u^*+\beta(1-u^*)}\right] + \frac{\Gamma}{(1-\gamma)\Lambda}\left[\frac{(\beta-\gamma\psi)v^*}{\alpha}+\frac{\eta(1-\gamma)(1-u^*)}{1-\eta}\right] + \alpha u^* + \frac{(1-\eta)v^*(1-v^*)}{1-u^*}\right\},$$

$$J_{33} = \frac{w^*}{\Gamma}\left\{\alpha(1-\eta)\left\{\frac{\gamma}{1-\gamma}\left(\frac{(1-\eta)v^*}{\alpha}+\frac{\eta(1-u^*)}{1-\eta}\right)\left[\psi+\theta-\frac{\beta}{(1-\eta)u^*+\beta(1-u^*)}\right]\right.\right.$$
$$\left.\left.+\Lambda\left(\frac{v^*(1-v^*)}{\alpha(1-u^*)}+\frac{u^*}{1-\eta}\right)\right\}+\frac{\Gamma s^*}{z^*}\right\},$$

$$\Gamma = \Lambda[\alpha(1-\eta)-\eta\beta]v-\frac{\gamma\alpha(1-\eta)}{1-\gamma}\left[\psi+\theta-\frac{\beta}{(1-\eta)u+\beta(1-u)}\right].$$

The determinant of the bordered Hessian of the Jacobean matrix is

$$BJ = \frac{-s^*w^*}{(1-\gamma)\Gamma z^*}\left\{\alpha\gamma\left[\psi+\theta-\frac{\beta}{(1-\eta)u^*+\beta(1-u^*)}\right]\left[(1-\eta)\left(1-\frac{1-\eta}{\sigma}\right)+\eta(u^*-\eta)\right]\right.$$
$$\left.+\left[\frac{M(u^*)}{u^*}\left(v^*-\frac{\alpha(1-u^*)}{1-\eta}\right)^{-1}\right][\eta(1-\gamma)+(\beta-\gamma\psi)]\right\}, \quad \text{(F.7)}$$

where

$$M(u^*) \equiv \eta\beta v^{*2}+\alpha^2u^{*2}+\left\{[\alpha(1-\eta)-\eta\beta]-\frac{\alpha[\alpha(1-\eta)-\eta(\beta-\gamma\psi)]u^*}{[\eta(1-\gamma)+\beta-\gamma\psi]v^*}\right\}$$
$$\times\left[v^*-\frac{\alpha(1-u^*)}{1-\eta}\right]v^*u^*.$$

We have shown that under Condition S, $BJ>$(*resp.* $<$)0 if $\Gamma<$ (*resp.* $>$) 0.

The determinant of the Jacobean (i.e., the product of the eigenvalues) is

$$DetJ = \frac{-\alpha(1-\eta)u^*s^*[(1-\tau)w^*]^2}{(1-\gamma)\Gamma z^*}\left\{\left(\frac{1}{\sigma}-\frac{1-u^*}{1-\eta}\right)\left[\psi+\theta-\frac{\beta}{(1-\eta)u+\beta(1-u)}\right]\frac{\gamma\eta}{u^*}\right.$$
$$\left.+\frac{\eta(1-\gamma)+(\beta-\gamma\psi)}{1-\eta}\right\}. \quad \text{(F.8)}$$

We have shown under Condition S, the terms in the large braces in $DetJ$ are positive and $DetJ>$ (*resp.* $<$)0 if $\Gamma<$ (*resp.* $>$)0. To investigate under what conditions $\Gamma<0$, we rewrite Γ as

$$\Gamma \equiv \{\Delta_s\Delta_p v^*\}+\left\{-\frac{\alpha(1-\eta)}{1-\gamma}\gamma\left[\psi+\theta-\frac{\beta}{(1-\eta)u+\beta(1-u)}\right]\right\}, \quad \text{(F.9)}$$

where $\Delta_p \equiv \alpha(1-\eta)-\eta\beta$ and

$$\Delta_s \equiv \left(\alpha+\frac{\alpha\gamma}{1-\gamma}\right)(1-\eta)-\eta\left(\beta+\frac{\beta\gamma}{1-\gamma}-\frac{\gamma\psi}{1-\gamma}\right)=\frac{1}{1-\gamma}[\alpha(1-\eta)-\eta(\beta-\gamma\psi)].$$

In order to obtain $\Gamma<0$, it requires sufficiently large values for γ, ψ and θ. Thus, given θ, the claims in Proposition 4 remain to hold true. To quantify the result, we conducted some simulations. We find that as long as θ is sufficiently large, local indeterminacy emerges in a reasonable space of (γ, ψ).

We demonstrate one numerical example. Appendix Table F1 and Appendix Figs. F1 and F2 are the results under $\theta = 0.3$. It is obvious to see that except for there being a trade-off among the degree of public service externality, the degree of

Table F1
Simulations ($\theta = 0.3$)

Variations	A	Minimum of ψ	Minimum of γ	Minimum of sector specific externalities $\left(\frac{\alpha\gamma}{1-\gamma}, \frac{\beta\gamma}{1-\gamma}\right)$	Minimum of negative aggregate externality $\frac{\psi\gamma}{1-\gamma}$	Minimum of the external effect of relative congestion $\frac{\theta\gamma}{1-\gamma}$	Region for indeterminacy
Benchmark[a]	0.023	0.57	0.125	(0.042, 0.100)	0.142	0.042	Appendix Figs. F1 and F2
$\rho = 0.01$	0.015	0.64	0.11	(0.037, 0.086)	0.123	0.037	Similar
$\rho = 0.05$	0.037	0.50	0.133	(0.046, 0.107)	0.153	0.046	Similar
$\sigma = 1.1$	0.018	0.51	0.131	(0.045, 0.105)	0.151	0.045	Similar
$\sigma = 3$	0.041	0.63	0.040	(0.093, 0.133)	0.133	0.040	Similar
$\alpha = 0.42(>\eta)$	0.018	0.83	0.322	(0.199, 0.275)	0.474	0.142	Similar
$\eta = 0.1(<\alpha)$	0.035	0.87	0.330	(0.147, 0.344)	0.492	0.147	Similar
$\eta = 0.3(=\alpha)$	0.015	0.35	0.010	(0.003, 0.007)	0.007	0.003	Similar
$\eta = 0.4(>\alpha)$	0.009	0.24	0.191	(0.007, 0.165)	0.159	0.070	Similar

[a]Benchmark parameters: $\tau = 0.2$, $\phi = 0.085$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$, $\theta = 0.3$.

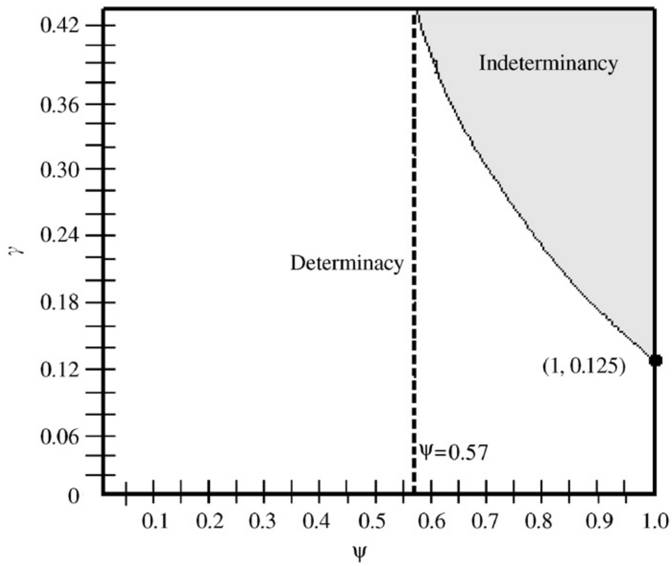

Fig. F1. $\theta = 0.3$; *Note*: (1) $A = 0.023$, $\tau = 0.2$, $\phi = 0.085$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$, $\theta = 0.3$. (2) γ represents the degree of externality of productive public spending and ψ represents the degree of congestion. (3) The shaded area presents the region of indeterminacy and $\psi = 0.57$ is the minimum of ψ.

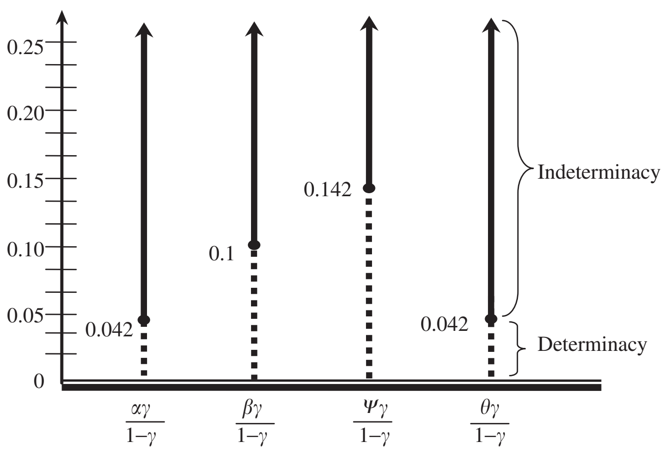

Fig. F2. Externalities ($\theta = 0.3$); *Note*: (1) $A = 0.023$, $\tau = 0.2$, $\phi = 0.085$, $\rho = 0.025$, $\sigma = 1.5$, $\alpha = 0.3$, $\eta = 0.2$, $\theta = 0.3$. (2) Values $\alpha\gamma/(1-\gamma)$ and $\beta\gamma/(1-\gamma)$ are the degree of sector-specific externalities associated with vK and uH, respectively; $\psi\gamma/(1-\gamma)$ is the degree of negative aggregate externality; $\theta\gamma/(1-\gamma)$ is the degree of the external effect of relative congestion. (3) The bold segment is for indeterminacy, and 0.042, 0.1, 0.142 and 0.042, respectively, are the minimal required value of each kind of externalities to establish local indeterminacy.

aggregate congestion and the degree of relative congestion, all other properties in the main text maintain to hold true in this model.

References

Aschauer, D.A., 1989. Is public expenditure productive? Journal of Monetary Economics 23, 177–200.

Barro, R.J., 1990. Government spending in a simple model of endogenous growth. Journal of Political economy 98, S103–S125.

Barro, R.J., Sala-i-Martin, X., 1992. Public finance in models of economic growth. Review of Economic Studies 59, 645–661.

Basu, S., Fernald, J.G., 1997. Returns to scale in US production: estimates and implications. Journal of Political Economy 205, 249–283.

Batten, D.F., Karlsson, C., 1996. Infrastructure and the Complexity of Economic Development. Springer, New York.

Benhabib, J., Meng, Q., Nishimura, K., 2000. Indeterminacy under constant returns to scale in multisector economics. Econometrica 68, 1541–1548.

Benhabib, J., Perli, R., 1994. Uniqueness and indeterminacy: on the dynamics of endogenous growth. Journal of Economic Theory 63, 113–142.

Ben-Gad, M., 2003. Fiscal policy and indeterminacy in models of endogenous growth. Journal of Economic Theory 108, 322–344.

Bond, E.W., Wang, P., Yip, C.K., 1996. A general two-sector model of endogenous growth with human and physical capital: balanced growth and transitional dynamics. Journal of Economic Theory 68, 147–173.

Burnside, C., 1996. Production function regressions, returns to scale, and externalities. Journal of Monetary Economics 37, 177–201.

Chen, B.-L., Lee, S.-F., 2005. Congestible public goods and indeterminacy in a two-sector endogenous growth model. IEAS Working Paper No. 05-A003, Academia Sinica, Taipei.

Edwards, J.H.Y., 1990. Congestion function specification and the 'publicness' of local public good. Journal of Urban Economics 27, 80–96.

Eicher, T., Turnovsky, S.J., 2000. Scale, congestion and growth. Economica 67, 325–346.

Fisher, W.H., Turnovsky, S.J., 1998. Public investment, congestion, and private capital accumulation. Economic Journal 108, 399–413.

Glomm, G., Ravikumar, B., 1994. Public investment in infrastructure in a simple growth model. Journal of Economic Dynamics and Control 18, 1173–1188.

Glomm, G., Ravikumar, B., 1997. Productive government expenditures and long-run growth. Journal of Economic Dynamics and Control 21, 183–204.

Gramlich, E.M., 1994. Infrastructure investment: a review essay. Journal of Economic Literature 76, 12–21.

Jones, L.E., Manuelli, R.E., Rossi, P.E., 1993. Optimal taxation in models of endogenous growth. Journal of Political economy 101, 485–517.

Kehoe, T., 1991. Computation and multiplicity of equilibria. In: Hildenbrand, W., Sonnenschein, H. (Eds.), Handbook of Mathematical Economics, vol. IV. pp. 2133–2134.

Lynde, C., Richmond, J., 1993a. Public capital and long-run costs in UK manufacturing. Economic Journal 103, 880–893.

Lynde, C., Richmond, J., 1993b. Public capital and total factor productivity. International Economic Review 34, 401–444.

McMillan, M.L., 1989. On measuring congestion of local public goods. Journal of Urban Economics 26, 131–137.

Meng, Q., Yip, C.K., 2005. On indeterminacy in a one-sector models of the business cycle with factor-generated externalities. Working Paper.

Mino, K., 2001. Indeterminacy and endogenous growth with social constant returns. Journal of Economic Theory 97, 203–222.

Mulligan, C., Sala-i-Martin, X., 1993. Transitional dynamics in two-sector models of endogenous growth. Quarterly Journal of Economics 108, 737–773.

Raurich, X., 2001. Indeterminacy and government spending in a two-sector model of endogenous growth. Review of Economic Dynamics 4, 210–229.

Rebelo, S., 1991. Long-run policy analysis and long-run growth. Journal of Political Economy 99, 500–521.

Shioji, E., 2001. Public capital and economic growth: a convergence approach. Journal of Economic Growth 6, 205–227.

Tanzi, V., Schuknecht, L., 2000. Public Spending in the 20th Century: A Global Perspective. Cambridge University Press, Cambridge.

Turnovsky, S.J., 1996. Optimal tax, debt, and expenditure policies in a growing economy. Journal of Public Economics 60, 21–44.

Turnovsky, S.J., 2000. Fiscal policy, elastic labor supply, and endogenous growth. Journal of Monetary Economics 45, 185–210.

Weder, M., 2001. Indeterminacy in a small open economy Ramsey growth model. Journal of Economic Theory 98, 339–356.