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Testing time reversibility without moment restrictions

Yi-Ting Chen[a], Ray Y. Chou[b], Chung-Ming Kuan[a,b,*]

[a]*Department of Economics, 21 Hsu-Chow Road, National Taiwan University, Taipei 10020, Taiwan*
[b]*Institute of Economics, Academia Sinica, Nankang, Taipei 115, Taiwan*

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Abstract

In this paper we propose a class of new tests for time reversibility. It is shown that this test has an asymptotic normal distribution under the null hypothesis and non-trivial power under local alternatives. A novel feature of this test is that it does not have any moment restriction, in contrast with other time reversibility and linearity tests. Our simulations also confirm that the proposed test is very robust when data do not possess proper moments. An empirical study of stock market indices is also included to illustrate the usefulness of the new test. © 2000 Published by Elsevier Science S.A. All rights reserved.

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1. Introduction

It is well known that Gaussian ARMA models are unable to capture many interesting dynamic features of data, such as asymmetric behaviors and clustering of volatility. Therefore, there has been a growing interest in nonlinear time-series models and non-Gaussian distributions; see e.g., Tong (1990) and

* Corresponding author. Department of Economics, 21 Hsu-Chow Road, National Taiwan University, Taipei 10020, Taiwan. Tel.: + 886-2-2351-9641 ext. 522; fax: + 886-2-2321-5704.
E-mail address: ckuan@ccms.ntu.edu.tw (C.M. Kuan)

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Granger and Teräsvirta (1993). Numerous diagnostic tests were also devised, e.g., Hinich (1982), Engle (1982), McLeod and Li (1983), Keenan (1985), Tsay (1986, 1989), Petruccelli and Davies (1986), Luukkonen et al. (1988), Lee et al. (1993), Benghabrit and Hallin (1996), and Wong (1997). These tests focus on either linearity or Gaussianity but not both and hence may yield misleading results. For instance, de Lima (1997) showed that a linearity test that ignores potential non-Gaussianity may perform quite poorly.

A stationary time series is said to be time reversible if its finite-dimensional distributions are all invariant to the reversal of time indices; otherwise, it is time irreversible. In particular, sequences of i.i.d. random variables and stationary Gaussian processes are time reversible. On the other hand, a linear, non-Gaussian process is time irreversible in general, except when its coefficients satisfy certain constraints; Tong (1990) also argued that: "time irreversibility is the rule rather than the exception when it comes to nonlinearity" (p. 197). See also Weiss (1975), Findley (1986), and Hallin et al. (1988). As such, a test of time reversibility may be viewed as a joint test of linearity and Gaussianity and can serve as a useful diagnostic check in model building, as noted in Cox (1981). Rejecting the null hypothesis suggests that Gaussian ARMA models are inappropriate; instead, nonlinear and/or non-Gaussian models should be considered. For empirical applications of such a test, see Rothman (1994) and Ramsey and Rothman (1996).

In this paper we propose a class of new tests for time reversibility. The proposed test is based on the implication that the differences of the series being tested have symmetric marginal distributions. By contrast, the test of Ramsey and Rothman (1996) focuses only on the third moment of these distributions. We show that this test has a limiting null distribution and nontrivial local power. A novel feature of this test is that it does not have any moment restrictions, whereas the tests of Ramsey and Rothman (1996) and Hinich and Rothman (1998) require finite sixth moment. Note that most of linearity tests also have similar requirements. Our simulations confirm that the proposed test is very robust when data do not possess proper moments. This feature makes the proposed test a useful complement to the existing diagnostic tests.

This paper proceeds as follows. In Section 2, we study some implications of time reversibility. In Section 3, we introduce the new test and analyze its asymptotic behavior. Test implementation is discussed in Section 4. Simulation results are reported in Section 5. An empirical study of stock indices is given in Section 6. Section 7 concludes the paper.

2. Implications of time reversibility

A (strictly) stationary time series $\{Y_t\}$ is said to be time reversible if its finite-dimensional distribution functions $F_{Y_{t_1},\dots,Y_{t_n}} = F_{Y_{t_n},\dots,Y_{t_1}}$, for any n-tuple

$t_1 < \cdots < t_n$; otherwise, it is time irreversible. In particular, time reversibility implies that for any $(a,b) \in \mathbb{R}^2$,

$$F_{Y_t,Y_{t-k}}(a,b) = F_{Y_t,Y_{t-k}}(b,a), \quad k = 1,2, \ldots .$$

Define $A(x) = \{(a,b): b-a \leqslant x\}$ and $B(x) = \{(a,b): b-a \geqslant -x\}$, where x is a real number. It is not difficult to see that for every x,

$$\int_{B(x)} \mathrm{d}F_{Y_t,Y_{t-k}}(a,b) = 1 - \int_{A(-x)} \mathrm{d}F_{Y_t,Y_{t-k}}(a,b),$$

$$\int_{A(x)} \mathrm{d}F_{Y_t,Y_{t-k}}(a,b) = \int_{B(x)} \mathrm{d}F_{Y_t,Y_{t-k}}(a,b).$$

Letting $X_{t,k} = Y_t - Y_{t-k}$, we have from time reversibility that

$$F_{X_{t,k}}(x) = \int_{A(x)} \mathrm{d}F_{Y_t,Y_{t-k}}(a,b) = 1 - \int_{A(-x)} \mathrm{d}F_{Y_t,Y_{t-k}}(a,b) = 1 - F_{X_{t,k}}(-x).$$

This proves the following implication of time reversibility.

Theorem 1. Let $\{Y_t\}$ be a time reversible process. Then for every $k = 1,2, \ldots,$ the distribution of $X_{t,k} = Y_t - Y_{t-k}$ is symmetric about the origin.

In practice, testing the distribution symmetry of $X_{t,k}$ for every k is infeasible. We may concentrate only on $X_{t,k}$ for $k = 1,2,\ldots,K$, where K is a small number. As an illustration, we simulate self-exciting threshold autoregressive (SETAR) processes:

$$Y_t = \begin{cases} \alpha_1 Y_{t-1} + \varepsilon_t & \text{if } Y_{t-1} \geqslant \delta, \\ \alpha_2 Y_{t-1} + \varepsilon_t & \text{if } Y_{t-1} \geqslant \delta, \end{cases}$$

with $(\alpha_1, \alpha_2, \delta) = (-0.5, 0.4, 1)$ and $(0.5, -0.4, 1)$, where ε_t are i.i.d. N(0, 1). These two processes were also studied by Rothman (1992) and will be referred to as SETAR-1 and SETAR-2, respectively. The simulated distributions of $X_{t,k}$, $k = 1,2,3$, based on 200,000 observations are shown in Fig. 1. It is visually clear that the distributions of $X_{t,1}$ are asymmetric. Therefore, both SETAR-1 and SETAR-2 are time irreversible.

While evaluating distribution shape may not be easy, it is relatively simpler to check related moments. Recall that the characteristic function of a symmetric distribution is real and even so that its imaginary part is zero; see e.g., Shiryayev (1995). Then for each $k = 1,2, \ldots, X_{t,k}$ has a symmetric distribution if, and only if,

$$h_k(\omega) := \mathbb{E}[\sin(\omega X_{t,k})] = 0$$

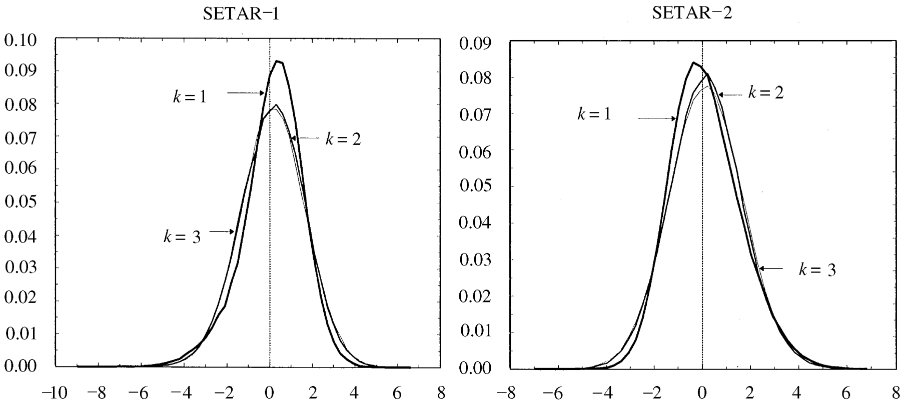

Fig. 1. Simulated marginal distributions of $\{X_{t,k}\}$ for $k = 1, 2, 3$.

for all $\omega \in \mathbb{R}^+$. As

$$\left.\frac{\partial^{2j+1} h_k(\omega)}{\partial \omega^{2j+1}}\right|_{\omega=0} = (-1)^j \mu_{2j+1,k}, \quad j = 0, 1, 2, \dots,$$

where $\mu_{i,k}$ is the ith moment of $X_{t,k}$. Time reversibility now implies all the odd moments of $X_{t,k}$, if exist, are zero. Observe that

$$\begin{aligned}\mu_{3,k} &= \mathbb{E}(Y_t^3) - 3\mathbb{E}(Y_t^2 Y_{t-k}) + 3\mathbb{E}(Y_t Y_{t-k}^2) - \mathbb{E}(Y_{t-k}^3) \\ &= -3\mathbb{E}(Y_t^2 Y_{t-k}) + 3\mathbb{E}(Y_t Y_{t-k}^2),\end{aligned}$$

by stationarity. Ramsey and Rothman (1996) suggested to test $\mathbb{E}(Y_t^2 Y_{t-k}) = \mathbb{E}(Y_t Y_{t-k}^2)$ which is equivalent to testing $\mu_{3,k} = 0$. One could, as Gupta (1967), directly test $\mu_{3,k}$ using its finite sample counterpart; see also Section 5.

The moment-based tests for time reversibility require higher-order moments of the process being tested to be finite. For example, the tests of Ramsey and Rothman (1996) and Hinich and Rothman (1998) are valid provided that the sixth moment exists. Jansen and de Vries (1991) and de Lima (1997) found, however, that the maximal moment exponents of many financial time series do not exceed four. The moment requirement therefore rules out many economic and financial time series. Even when higher-order moments exist, it is well known that $\mu_{3,k} = 0$ is not equivalent to distribution symmetry. In fact, Ord (1968) showed that a distribution may be asymmetric even when its odd moments are all zero; see also Li and Morris (1991).

3. New tests for time reversibility

We now propose a test aiming at distribution symmetry rather than moments. As discussed in the preceding section, we would like to test the hypothesis

$$h_k(\omega) = \mathbb{E}[\sin(\omega X_{t,k})] = 0 \tag{1}$$

for all $\omega \in \mathbb{R}^+$.

To construct a test of (1), we first introduce a weighting function and integrate out ω. Let g be a function such that $\int_0^\infty g(\omega)\,\mathrm{d}\omega < \infty$. Then under the null hypothesis,

$$\int_0^\infty h_k(\omega) g(\omega)\,\mathrm{d}\omega = 0 \tag{2}$$

for each k. Define the function ψ_g as

$$\psi_g(x) = \int_0^\infty \sin(\omega x) g(\omega)\,\mathrm{d}\omega. \tag{3}$$

By changing the order of integration, (2) is equivalent to

$$\mathbb{E}[\psi_g(X_{t,k})] := \int_{-\infty}^\infty \psi_g(x)\,\mathrm{d}F_{X_{t,k}}(x) = 0. \tag{4}$$

To test hypothesis (4), we employ its sample counterpart:

$$\bar{\psi}_{g,k} = \frac{1}{T-k} \sum_{t=k+1}^{T} \psi_g(x_{t,k}),$$

where $x_{t,k}$ are the observations of $X_{t,k}$. Because ψ_g is a static transformation, $\{X_{t,k}\}$ and $\{\psi_g(X_{t,k})\}$ are also stationary for each k. Then, under mild regularity conditions on the correlation structure of $\psi_g(X_{t,k})$, a central limit theorem holds:

$$\sqrt{T-k}\,(\bar{\psi}_{g,k} - \theta_{g,k})/\sigma_{g,k} \overset{A}{\sim} \mathrm{N}(0,1), \tag{5}$$

where $\theta_{g,k} = \mathbb{E}[\psi_g(X_{t,k})]$, and

$$\sigma_{g,k}^2 = \lim_{T\to\infty} \operatorname{var}\left(\frac{1}{\sqrt{T-k}} \sum_{t=k+1}^{T} \psi_g(X_{t,k})\right) = \operatorname{var}(\psi_g(X_{t,k}))$$
$$+ 2 \lim_{T\to\infty} \left(\sum_{\tau=1}^{T-k-1} \left(1 - \frac{\tau}{T-k}\right) \operatorname{cov}(\psi_g(X_{t,k}), \psi_g(X_{t-\tau,k})) \right).$$

We do not give specific conditions ensuring the central limit theorem; see Theorem 5.15 of White (1984) for more details. It is important to note that, because $\psi_g(X_{t,k})$ is bounded, no moment condition on Y_T is needed to ensure asymptotic normality.

In view of (5), the proposed test statistic is

$$\mathscr{C}_{g,k} = \sqrt{T-k}\,\bar{\psi}_{g,k}/\hat{\sigma}_{g,k}, \tag{6}$$

where $\hat{\sigma}^2_{g,k}$ is a consistent estimator for $\sigma^2_{g,k}$:

$$\hat{\sigma}^2_{g,k} = \frac{1}{T-k}\sum_{t=k+1}^{T}(\psi_g(x_{t,k}) - \bar{\psi}_{g,k})^2 + \frac{2}{T-k}\sum_{\tau=1}^{T-k-1}\kappa(\tau)\sum_{t=\tau+1}^{T}(\psi_g(\chi_{t,k}) - \bar{\psi}_{g,k})(\psi_g(x_{t-\tau,k}) - \bar{\psi}_{g,k}), \tag{7}$$

with κ a kernel function ensuring that $\hat{\sigma}^2_{g,k}$ is nonnegative. Estimators with different κ functions are studied by e.g., Newey and West (1987), Andrews (1991), and Politis and Romano (1994). The theorem below shows that the proposed test has an asymptotic normal distribution under the null and nontrivial local power against deviations from $\theta_{g,k} = 0$. This is a straightforward consequence of the asymptotic normality result (5).

Theorem 2. Given a stationary process $\{Y_t\}$, let $X_{t,k} = Y_t - Y_{t-k}$. Suppose that for each k, $\{\psi_g(X_{t,k})\}$ obeys a central limit theorem, where ψ_g is defined in (3). Then under the null hypothesis: $\theta_{g,k} = 0$,

$$\mathscr{C}_{g,k} \overset{A}{\sim} \mathrm{N}(0, 1);$$

under the local alternative hypothesis: $\theta_{g,k} = \delta/\sqrt{T-k}$, where δ is a nonzero constant,

$$\mathscr{C}_{g,k} \overset{A}{\sim} \mathrm{N}(\delta/\sigma_{g,k}, 1).$$

Remark 1. The proposed test is a general test for symmetry. This is a time-reversibility test for $\{Y_t\}$ if it is applied to the differenced series $\{X_{t,k}\}$. One may also apply the same test to $\{Y_t\}$ and determine whether this series has symmetric marginal distributions.

Remark 2. The proposed test does not require any moment conditions and hence has much wider applicability than moment-based tests. As a rival approach, the entropy-based test of Robinson (1991) can also be applied to test for symmetry and time reversibility. However, this test is valid under quite stringent conditions which rule out the normal and numerous leptokurtic distributions.

4. Implementing the new test

The proposed test $\mathscr{C}_{g,k}$ is a class of tests, depending on the weighting function g. An appropriate choice of g can enhance the power of this test. It is clear from (2) that g should be chosen such that h will not be integrated to zero when $X_{t,k}$ are asymmetric. Such choices of course depend on the unknown function h_k. In this section, we will first characterize the pattern of h_k and then establish a rule of thumb for choosing g.

Suppose for now that $F_{X_{t,k}}$ has the density function f_k. Then,

$$h_k(\omega) = \int_0^\infty \sin(\omega x)[f_k(x) - f_k(-x)\,\mathrm{d}x$$

$$= \frac{1}{\omega}\int_0^\infty \sin(z)[f_k(z/\omega) - f_k(-z/\omega)]\,\mathrm{d}z.$$

These expressions suggest than h_k should first grow from $h_k(0) = 0$ and then essentially decline to zero as ω tends to infinity. The following examples illustrate some patterns of h_k. Consider an exponentially distributed random variable that "centered" at its mean $\beta > 0$. It can be shown that

$$h_k(\omega) = \frac{\beta\omega\cos(\beta\omega) - \sin(\beta\omega)}{1 + (\beta\omega)^2}.$$

The h_k functions with different β are plotted in Fig. 2. One can easily see that these h_k functions are damping sine waves and that their amplitude and periodicity are determined by the value of β.

Consider also the random variable that has the standardized log-normal distribution with the asymmetry parameter $\lambda > 0$:

$$X = \frac{\exp(Z\lambda - \lambda^2/2) - 1}{(\exp(\lambda^2) - 1)^{1/2}}, \tag{8}$$

where $Z \sim \mathrm{N}(0, 1)$; see Johnson and Kotz (1970, p. 117). Although the resulting h_k function does not have an analytic form, we can simulate its behavior. For each $j = 1, \dots, R$, we generate random numbers $x_{t,j}$, $t = 1, \dots, T$, from the standardized log-normal distribution and compute the sample average:

$$\hat{h}_k^{(j)}(\omega) = \frac{1}{T}\sum_{t=1}^{T} \sin(\omega x_{t,j}).$$

The simulated h_k function is obtained by averaging R sample averages $\hat{h}_k^{(j)}$. We plot h_k with different λ in Fig. 3. They are similar to those functions in Fig. 2

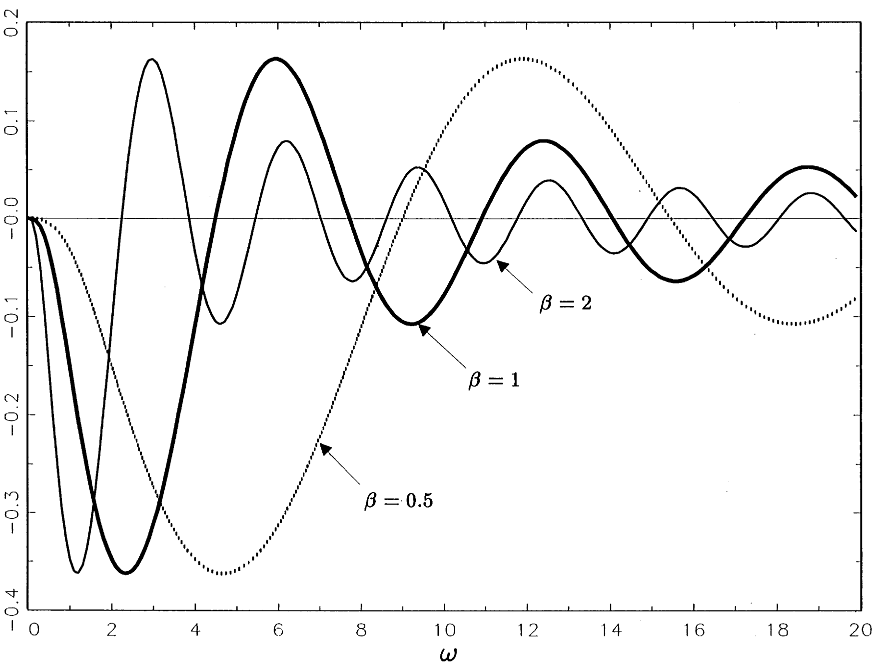

Fig. 2. $h(\omega)$ of the "centered" exponential distribution.

(damping sine waves). In brief, they all first increase (in magnitude) and eventually die out when ω approaches infinity.

In view of the patterns of h_k presented in Figs. 2 and 3, a rule of thumb is to choose a function that places more weights on small ω but much less weights on remote ω. Clearly, positive functions that are monotonically decreasing fit this purpose. Such a choice could also avoid h_k being integrated to zero when F_X is asymmetric. In particular, when g is chosen as a density function, then for each x, $\psi_g(x)$ is just the expected value of $\sin(\omega x)$ (with respect to g) and may have an analytic expression. This is particularly convenient for practitioners because $\psi_g(x)$ can be easily computed when the functional form of ψ_g is known. Therefore, leading choices of the weighting function are the density functions that are monotonically decreasing.

In this paper, we consider the following g functions so that the resulting ψ_g functions have analytic expressions. (1) the exponential density function with the parameter $\beta > 0$:

$$g(\omega) = \frac{1}{\beta}\exp\left(-\frac{\omega}{\beta}\right), \quad \omega > 0$$

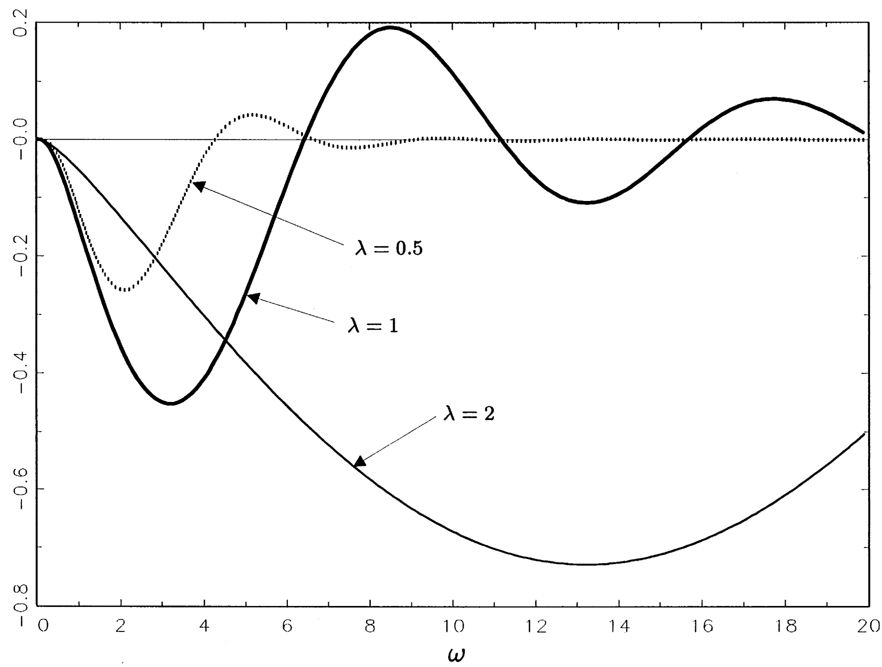

Fig. 3. Simulated $h(\omega)$ of the standardized log-normal distributions.

and (2) the half-normal density function:

$$g(\omega) = \frac{2}{\sqrt{2\pi}} \exp(-\omega^2/2).$$

It can be shown that in the former case,

$$\psi_g(x) = \frac{\beta x}{1 + (\beta x)^2} \tag{9}$$

and that in the latter case,

$$\psi_g(x) = -\mathrm{i}\,\mathrm{erf}\left(\frac{\mathrm{i}x}{\sqrt{2}}\right)\exp(x^2)^{-1/2}, \tag{10}$$

where $\mathrm{erf}(z) = (2/\sqrt{\pi})\int_0^z \exp(-s^2)\,\mathrm{d}s$ is the error function. The ψ_g function of (9) is very flexible; different statistics can be obtained by adjusting the parameter β. By inserting observations $x_{t,k}$ into (9) or (10), we can easily calculate $\bar{\psi}_{g,k}$ and hence the statistic $\mathscr{C}_{g,k}$. One may, of course, consider other g functions and compute the resulting ψ_g.

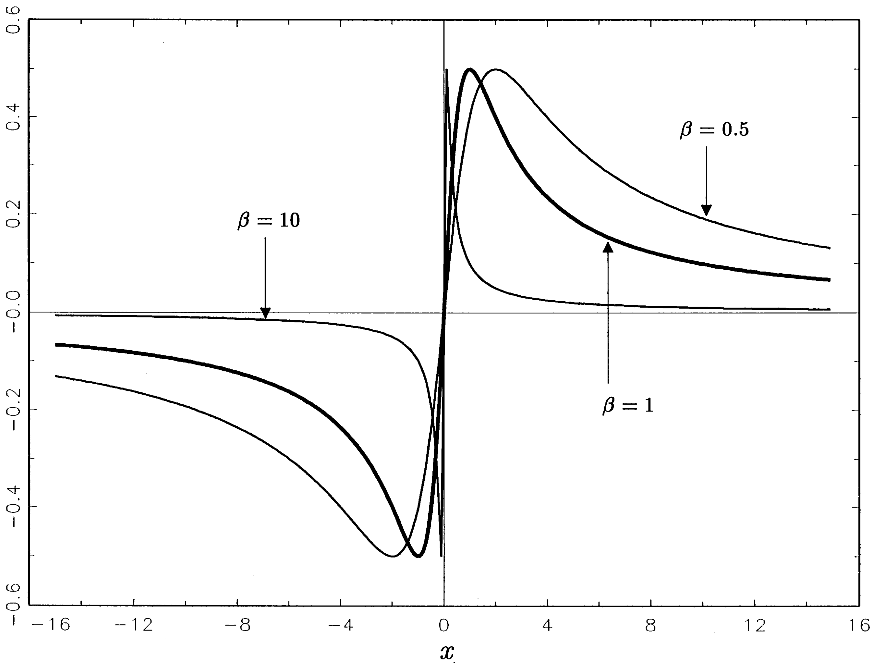

Fig. 4. ψ_g where g is the exponential density function.

It is also important to understand how the parameter β in (9) affects the performance of the proposed test. We plot the ψ_g functions with g the exponential weighting function and $\beta = 0.5$, 1, and 10 in Fig. 4. When β is large, ψ_g is very concentrated around $x = 0$ and declines very rapidly as x deviates from zero. This suggests that if the distribution of $X_{t,k}$ is widely dispersed, our test with a large β may not have good power because it is not very sensitive to the behavior of large x. For a more concentrated distribution, our test with a small β may place too much weight on large x. Unfortunately, we do not know what β value best fits our needs in practice. One way to circumvent this problem is to "standardize" the observations using their sample standard deviation and choose the weighting function with a moderate β, say, $\beta = 1$. In view of (9), this amounts to choosing β as the reciprocal of the sample standard deviation of $X_{t,k}$.

5. Monte Carlo simulations

In this section, we investigate the finite-sample performance of the proposed test $\mathscr{C}_{g,k}$ by simulations. We employ the exponential weighting function ($g = \exp$) with $\beta = 0.5, 1, 2, 10$ and the half-normal weighting function ($g = \text{hn}$);

these tests will be denoted as $\mathscr{C}_{\exp,k}$ and $\mathscr{C}_{\mathrm{hn},k}$, respectively. Note that we did not standardize data in simulations and hence can assess the effects of different β values. To compute the estimate $\hat{\sigma}^2_{g,k}$ in (7), we adopt the kernel function

$$\kappa(\tau)=\left(1-\frac{\tau}{T-k}\right)[1-c(T-k)^{-1/3}]^{\tau}+\frac{\tau}{T-k}[1-c(T-k)^{-1/3}]^{T-k-\tau},$$

where $c=0.5$, and follow the stationary bootstrap method of Politis and Romano (1994). Tests based on other κ functions, including the Bartlett kernel and the quadratic spectrum kernel of Andrews (1991), were also considered, but they did not perform well. In our simulations, the sample sizes are $T=100$ and 500, and the number of replications is 1000.

For comparison, we compute the third-moment-based test

$$\mathscr{M}_k=\sqrt{T-k}\, m_{3,k}/\hat{v}_{T,k},$$

where $m_{3,k}=(T-k)^{-1}\sum_{t=k+1}^{T}x^3_{t,k}$ is the sample counterpart of $\mu_{3,k}$, and $\hat{v}^2_{T,k}$ is a consistent estimator for

$$v_k^2=\mathrm{var}(X^3_{t,k})+2\lim_{T\to\infty}\left(\sum_{\tau=1}^{T-k-1}\left(1-\frac{\tau}{T-k}\right)\mathrm{cov}(X^3_{t,k},X^3_{t-\tau,k})\right),$$

computed as $\hat{\sigma}^2_{g,k}$ discussed before. Under the null hypothesis that $\mu_{3,k}=0$, $\mathscr{M}_k$ has the standard normal distribution asymptotically. We also considered the test of Ramsey and Rothman (1996). In some experiments (e.g., i.i.d. sequences of stable innovations), their variance formula may result in negative estimates. We therefore simulate their test based on the standard deviation of the simulated distribution, as suggested by Ramsey and Rothman. These two tests perform quite similarly; in fact, they are asymptotically equivalent, as discussed in Section 2. For simplicity, we reported only the results of $\mathscr{M}_k$; other results are available upon request.

In the simulation of empirical sizes, the nominal size is 5%. We consider i.i.d. sequences with the innovations generated from N(0, 1), Student's $t(3)$, stable distributions (the characteristic exponent $\alpha=1$ and the symmetry parameter $\delta=0, 1$). Note that $t(3)$ does not have finite third moment and that stable distributions with $\alpha=1$ do not have finite variance. Moreover, a stable distribution is symmetric if $\delta=0$, and it becomes more asymmetric when δ increases. Stable distributions are generated according to Chambers et al. (1976). We also generate data from a Gaussian AR(1) process

$$y_t=0.5y_{t-1}+\varepsilon_t$$

and a Gaussian MA(1) process

$$y_t=-0.5\varepsilon_{t-1}+\varepsilon_t,$$

where ε_t are i.i.d. N(0, 1). These results are summarized in Table 1.

Table 1
Empirical sizes of $\mathscr{M}_k$ and $\mathscr{C}_{g,k}$ when the nominal sizes are 5%

Process	k	$\mathscr{M}_k$ T = 100	500	$\mathscr{C}_{\exp,k,\beta=0\cdot5}$ 100	500	$\mathscr{C}_{\exp,k,\beta=1}$ 100	500	$\mathscr{C}_{\exp,k,\beta=2}$ 100	500	$\mathscr{C}_{\exp,k,\beta=10}$ 100	500	$\mathscr{C}_{\text{hn},k}$ 100	500
i.i.d. N(0, 1)	1	4.0	4.1	1.5	1.6	5.6	4.3	5.9	5.2	7.2	6.0	4.1	4.4
	2	1.8	3.5	0.2	0.5	3.0	2.8	6.4	5.6	5.3	5.2	2.4	3.8
	3	1.5	2.1	0.0	0.5	2.0	2.6	5.6	4.9	5.8	4.4	2.1	1.9
i.i.d. $t(3)$	1	0.9	1.0	3.5	3.2	5.8	5.9	6.7	5.4	8.4	5.6	6.3	4.9
	2	0.6	0.2	0.8	0.7	3.2	4.9	5.8	6.6	7.5	6.8	3.8	4.5
	3	0.1	0.0	1.0	0.7	2.0	3.3	5.7	5.8	7.1	4.8	3.0	4.6
i.i.d., stable	1	0.0	0.0	3.9	4.1	7.8	5.3	7.6	5.9	8.2	5.3	7.0	6.0
$\alpha = 1$	2	0.0	0.0	2.4	3.6	6.3	4.6	7.2	5.1	5.8	5.8	6.1	5.6
$\delta = 0$	3	0.0	0.0	1.4	1.8	5.5	4.0	7.5	5.0	7.2	3.7	4.3	4.6
i.i.d., stable	1	0.0	0.0	3.8	4.2	7.2	5.4	8.8	6.4	8.0	5.7	8.6	6.7
$\alpha = 1$	2	0.0	0.0	4.3	3.8	5.0	4.6	6.5	4.1	7.8	6.3	4.9	4.2
$\delta = 1$	3	0.0	0.0	2.7	2.6	2.9	4.8	6.2	5.7	9.6	5.7	5.3	4.5
Gaussian AR (1)	1	2.8	3.4	0.4	0.6	3.1	3.2	4.8	5.1	7.5	6.3	1.6	2.4
	2	0.5	2.1	0.0	0.1	2.0	2.2	4.2	3.6	8.0	5.0	1.2	1.8
	3	0.1	1.7	0.0	0.0	1.4	1.7	3.4	3.9	8.3	5.1	0.9	1.3
Gaussian MA (1)	1	1.7	3.5	1.8	2.6	6.9	5.4	6.5	5.9	8.1	6.0	4.9	5.5
	2	3.1	3.8	1.1	2.2	5.0	3.4	8.0	6.1	8.9	6.9	4.7	4.2
	3	1.9	2.6	0.7	1.3	4.3	3.9	5.6	5.7	7.7	5.9	4.2	3.9

Note: The entries are rejection frequencies in percentages.

It can be seen that the performance of $\mathscr{C}_{\exp,k}$ depends on β. When $\beta = 1$ or 2, the empirical sizes are close to the nominal size. For $\beta = 0.5$, the empirical sizes are usually very small (close to zero); for $\beta = 10$, empirical sizes are greater than the nominal size, especially when $T = 100$. These suggest that properly choosing β is important in determining the performance of $\mathscr{C}_{\exp,k}$. For the half-normal weighting function, the empirical sizes are also close to the nominal size. On the other hand, the $\mathscr{M}_k$ test usually has greater size distortion than $\mathscr{C}_{g,k}$. When data do not possess proper moments, i.e., i.i.d. with $t(3)$ and stable distributions, the empirical sizes of $\mathscr{M}_k$ are all close to zero, but those of the proposed tests are not.

For power simulations, we generate SETAR-1 and SETAR-2 as discussed in Section 2. We also generate non-Gaussian MA(1) processes with the standardized log-normal innovations ($\lambda = 1, 2$) and stable innovations ($\alpha = 1$, $\delta = 1$). Note that the log-normal distribution has finite moments of all orders, but its moments grow very rapidly. These results are summarized in Table 2.

For well-behaved nonlinear processes such as SETAR-1 and SETAR-2, $\mathscr{M}_k$ performs quite well, and $\mathscr{C}_{\text{hn},k}$ has comparable power. Although the $\mathscr{C}_{\exp,k}$ tests are not as good as $\mathscr{M}_k$ in these cases, they still have reasonable power.

Table 2
Empirical powers of $\mathscr{M}_k$ and $\mathscr{C}_{g,k}$ when the nominal sizes are 5%

Process	k	$\mathscr{M}_k$		$\mathscr{C}_{\exp,k,\beta=0.5}$		$\mathscr{C}_{\exp,k,\beta=1}$		$\mathscr{C}_{\exp,k,\beta=2}$		$\mathscr{C}_{\exp,k,\beta=10}$		$\mathscr{C}_{\text{hn},k}$	
		T = 100	500	100	500	100	500	100	500	100	500	100	500
SETAR-1	1	32.0	98.2	22.7	96.4	29.5	89.5	26.8	74.2	31.8	85.1	33.1	96.2
	2	1.5	5.9	5.0	3.6	3.2	4.2	5.9	6.1	7.2	10.3	4.5	8.1
	3	0.7	1.9	2.0	0.2	2.2	2.9	5.6	4.5	5.9	5.6	2.0	2.9
SETAR-2	1	44.5	99.7	18.9	95.5	33.9	96.4	31.2	82.7	29.7	87.0	31.7	94.6
	2	1.4	3.0	0.8	1.2	3.7	4.3	5.4	4.6	6.8	7.5	3.3	5.6
	3	0.7	1.4	0.2	0.2	2.4	3.1	4.0	4.1	6.0	6.7	2.0	1.8
MA(1)	1	19.4	64.0	70.3	100.0	81.8	100.0	72.1	100.0	32.2	85.1	81.2	100.0
log-normal	2	0.0	0.1	0.4	0.5	1.1	2.3	4.7	3.5	6.8	6.3	1.1	2.3
$\lambda = 1$	3	0.2	0.1	0.2	0.3	1.5	2.7	4.0	4.4	8.0	6.5	1.0	1.7
MA(1)	1	5.6	16.9	13.3	99.9	53.3	100	89.9	100	97.8	100	28.0	99.7
log-normal	2	0.0	0.0	0.0	0.0	0.1	0.0	0.1	0.0	2.8	3.0	0.0	0.0
$\lambda = 2$	3	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.5	1.8	2.2	0.0	0.0
MA (1), stable	1	4.1	7.6	64.8	99.9	44.8	96.1	26.8	75.4	14.4	22.2	33.5	90.4
$\alpha = 1$	2	0.0	0.0	4.8	5.0	8.5	5.1	8.6	5.0	7.3	5.6	8.1	5.5
$\delta = 1$	3	0.0	0.0	5.5	4.0	6.6	4.8	6.7	5.4	7.6	6.5	6.7	4.6

Note: The entries are rejection frequencies in percentages.

Rothman (1992) found that the test of Ramsey and Rothman (1996) is more powerful than the bispectrum and BDS tests for SETAR-1 and SETAR-2. The empirical powers reported in Rothman's paper are indeed very close to those of $\mathscr{M}_k$. Hence, it is reasonable to believe that $\mathscr{C}_{\text{hn},k}$ also compares favourably with the bispectrum and BDS tests. When data are generated as an MA(1) process with log-normal innovations, the power of $\mathscr{M}_k$ drops very quickly as the asymmetry parameter λ increases, whereas $\mathscr{C}_{\exp,k}$ and $\mathscr{C}_{\text{hn},k}$ overwhelmingly dominate $\mathscr{M}_k$. In this case, it is quite remarkable that $\mathscr{C}_{\exp,k}$ with $\beta = 1$ and 2 have very high power even when $T = 100$. For the MA(1) process with stable innovations, $\mathscr{M}_k$ again has no power, whereas $\mathscr{C}_{\exp,k}$ with a small β has much better power performance.

These simulations confirm that the proposed test is quite robust to the moment property of the process being tested. Although there is no test that uniformly dominates other tests, our results suggest that $\mathscr{C}_{\exp,k}$ with $\beta = 1$ and 2 and $\mathscr{C}_{\text{hn},k}$ have quite good performance in all cases considered. Therefore, these tests can complement the $\mathscr{M}_k$ test or that of Ramsey and Rothman (1996) in practice.

Finally, we observe that asymmetry can be detected only for $k = 1$ in the power simulations. This is, however, due to other simulation designs. To see this, consider an MA(q) process $\{Y_t\}$ generated by some i.i.d. innovations. Clearly, Y_t and Y_{t-k} are independent for $k > q$ but dependent otherwise. Then for $k > q$,

$$\begin{aligned}\mathbb{E}[\sin(\omega X_{t-k})] &= \mathbb{E}[\sin(\omega Y_t)\cos(\omega Y_{t-k})] - \mathbb{E}[\cos(\omega Y_t)\sin(\omega Y_{t-k})] \\ &= \mathbb{E}[\sin(\omega Y_t)]\mathbb{E}[\cos(\omega Y_{t-k})] - \mathbb{E}[\cos(\omega Y_t)]\mathbb{E}[\sin(\omega Y_{t-k})] \\ &= 0,\end{aligned}$$

but it need not be zero for $k \leqslant q$. This shows that $X_{t,k}$ may still exhibit asymmetry for $2 \leqslant k \leqslant q$. In our simulations of MA(1) data, it is therefore not surprising to see asymmetry only for $k = 1$. for the SETAR process with the threshold variable Y_{t-d}, $d \geqslant 2$, we may also expect to see asymmetry for some $k \geqslant 2$.

6. An empirical example

In this section, we apply the proposed tests to study the rate of return of various stock market indices. Our data are taken from Taiwan Economic Data Center and contain six market indices from January 1, 1990 through May 31, 1997. The indices are the Dow Jones Industrial Average Index (DJIA) of New York, Credit Suisse Stock Index (CSS) of Zurich, Commerzbank Index (CB) of Frankfurt, Nikkei Dow Jones Index (NDJ) of Tokyo, Hang Seng Index (HS) of Hong Kong, and Weighted Stock Index (WS) of Taipei. For each index η_t, its rate of return is $Y_t = 100 \times (\log \eta_t - \log \eta_{t-1})$. If Y_t is time irreversible, $\log \eta_t$ would be neither a random walk nor a Gaussian ARIMA process. If so, one may want to model Y_t using, say, nonlinear models.

Table 3 gives the summary statistics and Ljung-Box Q statistics (with 50 lags) for the rates of return of these indices. Let $\alpha = \sup_{i>0} \mathbb{E}|Y|^i < \infty$ denote the maximal moment exponent of $\{Y_t\}$. In accordance with Hall (1982) and de Lima (1997), we also compute the estimates of α:

$$\hat{\alpha} = \left(R^{-1} \sum_{j=1}^{R} \log y_{T,T-j+1} - \log y_{T,T-R} \right)^{-1}, \tag{11}$$

where $y_{T,1} < y_{T,2} < \dots < y_{T,T}$, and $y_{T,R}, \dots, y_{T,T}$ in (11) are 10% right-tail observations. Choosing 10% left-tail observations yields similar estimate $\hat{\alpha}$. Strictly speaking, $\hat{\alpha}$ can only be computed for independent data; these estimates nevertheless still provide us information about existing moments. From Table 3, the sample (excess) kurtosis coefficients suggest that all the distributions have fat tails, especially the rate of return of CB. We can see that Ljung-Box's Q test cannot reject the null hypothesis of white noise for DJIA and HS at 5% level. According to $\hat{\alpha}$, all the series might not even have the third moment.

We compute $\mathscr{M}_k$ and $\mathscr{C}_{\exp,k}$. We follow the suggestion in Section 4 and set β as the reciprocal of the sample standard deviation of each series. In Fig. 5, we plot

Table 3
Some statistics of the rates of return of market indices

Index	T	Mean	Variance	Skewness	Kurtosis	Q (50)	$\hat{\alpha}$
DJIA	1873	0.051	0.608	− 0.093	2.817	60.271	2.756
CSS	1819	0.028	0.816	− 0.703	6.898	68.541[a]	2.852
CB	1843	0.026	1.523	− 0.808	37.468	84.931[b]	2.377
NDJ	1827	− 0.036	2.243	0.404	4.562	76.416[b]	2.381
HS	1841	0.090	1.869	− 0.468	4.757	66.881	2.720
WS	2136	0.009	4.261	0.073	5.139	146.38[b]	2.175

[a]Denote significance at 5% level.
[b]Denote significance at 1% level.
Note: T is the number of observations; Q (50) is the Ljung-Box statistic based on 50 lags; $\hat{\alpha}$ is the estimate of the maximal moment exponent.

these test statistics for $k = 1, \dots, 50$; the horizontal solid and dash lines represent the critical values at 5% and 1% level, respectively. It can be seen that $\mathscr{M}_1$ is significantly different from zero (rejects the null hypothesis) for DJIA and HS at 5% level and for NDJ at 1% level. Thus, one may believe that DJIA, NDJ, and HS are time irreversible. Note that $\mathscr{M}_2$ are also significantly different from zero for NDJ and HS at 5% level. For the remaining three series (CSS, CB and WS), all $\mathscr{M}_k$, are not significantly different from zero at 5% level. On the other hand, $\mathscr{C}_{\exp,1}$ rejects the null hypothesis for HS at 5% level and for DJIA, CSS and NDJ at 1% level, and $\mathscr{C}_{\exp,2}$ also rejects the null hypothesis for CSS, NDJ, and WS at 5% level. Therefore, the proposed test enables us to identify two more series (CSS and WS) that are time irreversible. The rates of return of CB is the only process that neither $\mathscr{M}_k$ nor $\mathscr{C}_{\exp,k}$ can reject the null.

It is interesting to note that the results of the Ljung-Box test and the time-reversibility test may have different implications for time-series models. The former shows that there is no evidence against the hypotheses that DJIA and HS are white noise series, whereas the latter suggests that they are time irreversible and should be handled by more complex models. For CB, the Ljung-Box test indicates that its rate of return is not a white noise and hence cannot be an i.i.d. sequence, but the proposed test suggests that it is time reversible. As the rate of return of CB has a very large (excess) kurtosis coefficient, the Lung-Box test results may not be reliable. If so, we still cannot rule out the possibility that the rate of return of CB is an i.i.d. sequence. More definite conclusion cannot be drawn until further tests are conducted. For example, we may also apply the proposed test to the rates of return (rather than their differences) and check whether their marginal distributions are symmetric. Such results may provide guidance for further modeling.

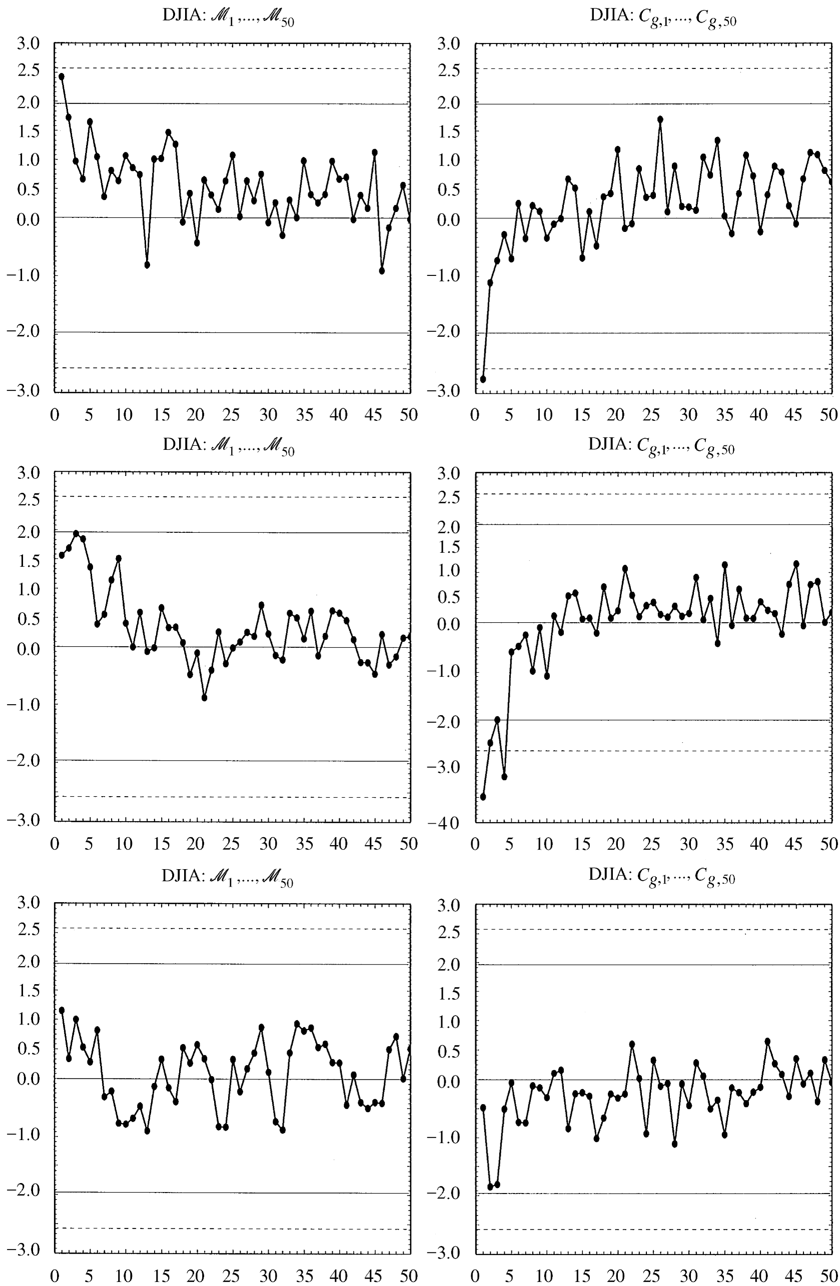

Fig. 5. The time-reversibility test results of the rates of return processes.

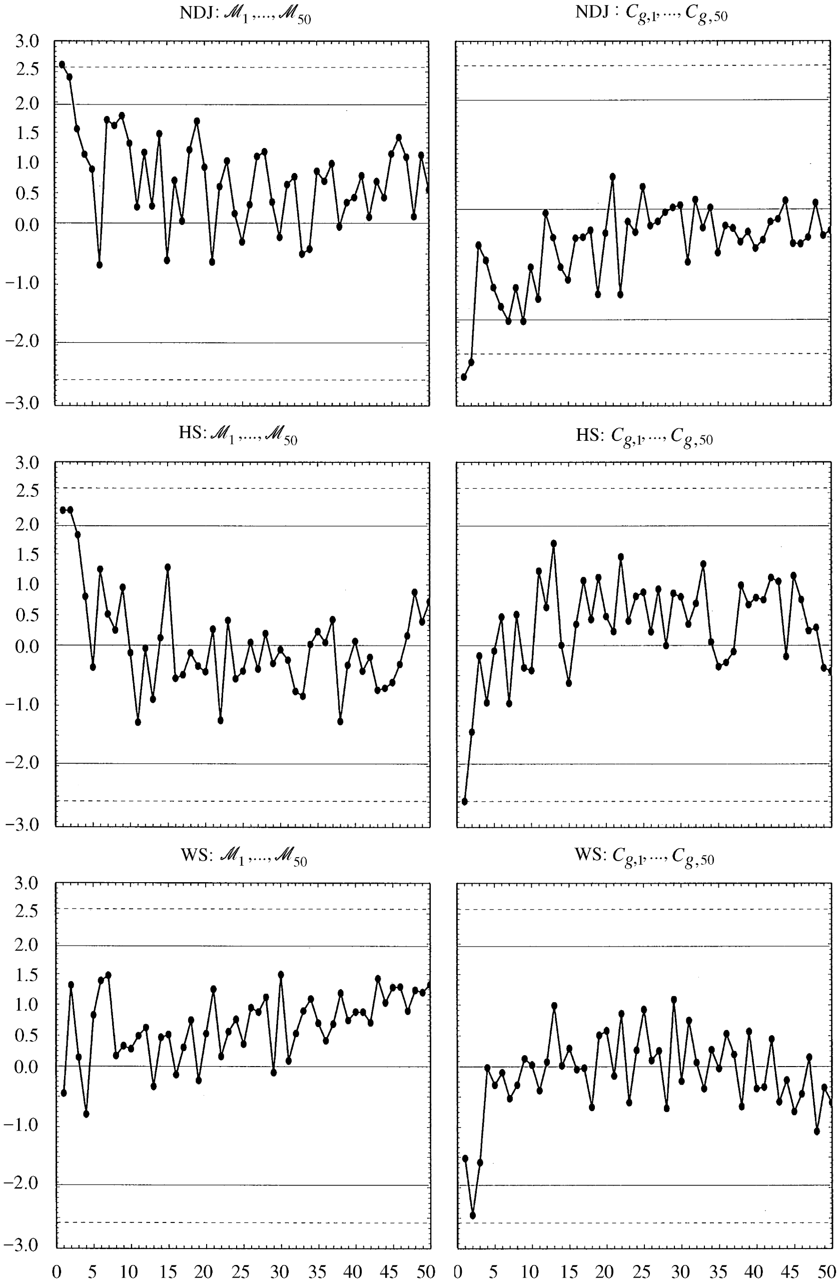

Fig. 5. (Continued).

7. Conclusions

Testing for time reversibility is important for model building in the time-series analysis. Once a time series is identified to be time irreversible, it is neither an i.i.d. sequence nor a Gaussian ARMA process. More efforts must be made to properly model the variables of interest. One may want to consider nonlinear models and/or non-Gaussian distributions. While more and more empirical evidences suggest that many economic and financial time series are highly leptokurtic or may even have infinite variance, most of the existing diagnostic tests for time reversibility and linearity are not really applicable because they require finite fourth moment (or moment of an even higher order).

In this paper, we propose a class of new tests for time reversibility and suggest different ways to implement it. The proposed test can be easily calculated and has an asymptotic normal distribution under the null. We also demonstrate that it is robust to processes that do not possess proper moments. The feature of robustness makes the new test a useful complement to the existing diagnostic tests, as shown in our simulations and empirical study. As the proposed test is in fact a test for distribution symmetry, it can also be applied when the distribution shape of the data is of interest. A drawback of our test is that it can only test the differenced series $\{X_{t,k}\}$ for each k. To jointly test $X_{t,k}$ for a collection of k values, a portmanteau test is needed. Extending the proposed test to a joint test is quite challenging because we must deal with the covariances among individual statistics. This is a direction for further research.

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