Asymmetric Contests with Initial Probabilities of Winning

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Abstract

We study contests in which players each have an initial probability of winning a prize, and they compete with one another by expending irreversible effort to win the prize. First, we consider two models in which the impact parameter is exogenous: the one with the logit-form function and the one with the all-pay-auction selection rule. We find that neither the equilibrium number of active players nor their identities nor the equilibrium (expected) effort levels of the players depend on the players' initial probabilities of winning. We find also that the possibility that the winner is determined by the players' initial probabilities of winning reduces prize dissipation, and tends to make most players better off, compared to the contest without this possibility. Then, we consider an extended model in which the impact parameter is endogenous. Interestingly, we find that every player may expend zero effort in equilibrium.

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Keywords: Asymmetric contest; Initial probability of winning; No-effort equilibrium; Prize dissipation; More efficient rent seeking

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1. Introduction

This paper studies situations in which players are vested with their initial probabilities of winning a prize, and compete with one another by expending irreversible effort to win the prize. Such contests with initial probabilities of winning are readily observed, but to the best of our knowledge, have not previously been studied. Each player's initial probability of winning may reflect customs, institutional rules, her natural talent, her educational background, or her previous efforts in the contest or other related contests. For example, in a rent-seeking contest, it may reflect her relationship with government officials established by her previous rent-seeking efforts. In an R&D contest, it may reflect her relevant knowledge and experience gained from her previous R&D efforts. Finally, in a sporting contest, it may reflect her natural talent or sporting skills (or ability) gained from her previous sporting efforts.

This paper formally studies the following simultaneous-move game. At the beginning of the game, the players know their valuations for the prize, their initial probabilities of winning the prize, and the information regarding the value of an "impact parameter" – the parameter which determines how much impact the players' initial probabilities of winning have on their final probabilities of winning. Next, they expend their effort simultaneously and independently in order to win the prize. Finally, the prize is awarded to one of the players. In this paper, the players are assumed to have different valuations for the prize and different initial probabilities of winning the prize.

First, this paper considers two models in which the impact parameter is exogenous: the one with the simplest logit-form function and the one with the all-pay-auction selection rule. In the former model, the simplest logit-form function is used in specifying each player's contest success function, which is a rule that describes the relationship among her initial probability of winning, the players' effort levels, and her final probability of winning. On the other hand, in the latter model, the selection rule of all-pay auctions is used in specifying each player's contest success function. Then, it considers an extended model in which the impact parameter is endogenous. This extended model is the same as the model with the simplest logit-form function
above with the exception that, in the extended model, the impact parameter is a nonincreasing function of the total effort level of the players.

This paper shows in the first two models that the equilibrium number of active players – that is, the number of players who expend positive effort in equilibrium – and their identities depend neither on the players' initial probabilities of winning the prize nor on the impact parameter. This follows from two facts. First, the equilibrium (expected) effort levels of the players are equal to those obtained in a reduced contest in which the same players compete to win the prize in the absence of initial probabilities of winning and their valuations for the prize are reduced proportionately by a unique factor determined by the value of the impact parameter. Second, any proportionate decrease in the players' valuations do not change their decisions on whether to be active or inactive.

Next, this paper shows in the first two models that the equilibrium (expected) effort levels of the players do not depend on the players' initial probabilities of winning the prize. However, the possibility that the winner is determined by the players' initial probabilities of winning – in other words, the possibility that the players' efforts are wasted – reduces prize dissipation (or rent dissipation in a rent-seeking contest), compared to the contest without this possibility.\(^3\) Interestingly, prize dissipation reduces only because of that possibility, but not because of the players' initial probabilities of winning per se. All these follow from two facts. First, the equilibrium (expected) effort levels of the players are equal to those obtained in a reduced contest in which the same players compete to win the prize in the absence of initial probabilities of winning and their valuations for the prize are reduced proportionately by a unique factor determined by the value of the impact parameter. Second, any proportionate decrease in the players' valuations reduces the equilibrium (expected) effort levels of the active players proportionately. More specifically, in the model with the simplest logit-form function, a decrease in each active player's valuation decreases her marginal gross payoff at every effort level while her marginal cost remains unchanged, so that her equilibrium effort level decreases.
Next, this paper shows in the first two models that the possibility that the winner is determined by the players' initial probabilities of winning tends to make better off most players with positive initial probability of winning, compared to the contest without this possibility. Intuitively, this follows from two facts. First, the possibility described above mitigates the competition among the players, which decreases the equilibrium (expected) effort levels of the active players. Second, given that possibility, the inactive players with positive initial probability of winning remain in the contest and have positive expected payoffs.

Finally, this paper shows that every player may expend zero effort in equilibrium, in the extended model in which the impact parameter is a nonincreasing function of the total effort level of the players. This result contrasts with the received result – that the no-effort equilibrium does not exist – in the literature on the theory of contests. To the best of our knowledge, the existence of the no-effort equilibrium is shown only in Baik (1998), which studies contests with difference-form contest success functions. Another interesting result in the extended model is that there may exist multiple pure-strategy Nash equilibria even though the simplest logit-form function is used in specifying each player's contest success function. This result is interesting because it is well known in the literature on the theory of contests that the Nash equilibrium is unique in contests with the simplest logit-form functions: See, for example, Cornes and Hartley (2005) and Yamazaki (2008).

This paper is closely related to Hillman and Riley (1989). They study both transfer and rent-seeking contests in which the players have different valuations for the prize. We extend their models by incorporating the players' initial probabilities of winning.

This paper is also related to Ansink (2011) and Faith et al. (2008). Ansink (2011) studies resource contests in which the players have exogenously given claims on shares of a contested resource; they compete with one another by exerting effort to secure larger shares of the resource; and their claims and effort levels jointly determine, through a contest-bankruptcy rule, their final shares of the resource. Faith et al. (2008) study intergenerational-transfer contests in which the parents first choose and commit to their children's bequest shares, and then given their
"claims" on bequest shares, the children compete with each other by expending effort to secure larger amounts of bequests and gifts.

Tullock (1980) and Hillman and Riley (1989) study rent-seeking contests in which the players expend their effort simultaneously, and establish that "efficient rent seeking" occurs – that is, rent dissipation is less than complete. Subsequently, Baik and Shogren (1992) and Leininger (1993) study endogenous timing in lopsided contests, and show that the endogenous timing of moves leads to "more efficient rent seeking" because, in equilibrium, the underdog moves first and restrains herself in order to avoid stiff competition against the favorite, which in turn allows the favorite to ease up and respond efficiently. On the other hand, the present paper shows that the presence of the players' initial probabilities of winning, or rather the possibility that the winner is determined by the players' initial probabilities of winning, leads to more efficient rent seeking even when the players move simultaneously.

There exist many papers that study asymmetric contests: See, for example, Morgan (2003), Baik (2004), Nti (2004), Stein and Rapoport (2004), Malueg and Yates (2005), Cornes and Hartley (2005), Siegel (2010), and Franke et al. (2013). A striking difference between this paper and the previous papers is that this paper incorporates the players' initial probabilities of winning into their contest success functions, whereas the previous papers do not.

The paper proceeds as follows. Section 2 develops the model with the simplest logit-form function in which the impact parameter is exogenous, and sets up the simultaneous-move game. In Section 3, we solve for the Nash equilibrium of the game, and obtain the equilibrium number of active players and their identities, the equilibrium effort levels of the players, and the equilibrium expected payoffs for the players. In Section 4, we examine the effects of changing the parameters on these outcomes of the game. Section 5 considers the model with the all-pay-auction selection rule in which the impact parameter is exogenous. Section 6 considers an extended model in which the impact parameter is endogenous. Finally, Section 7 offers our conclusions.
2. The model with the simplest logit-form function

Consider a contest in which $n$ risk-neutral players, 1 through $n$, each want to win a prize, and each player has an initial probability of winning the prize, where $n \geq 2$. The players compete with one another by expending irreversible effort to win the prize. The (final) probability that a player wins the prize is increasing in her own effort level, *ceteris paribus*, and decreasing in the rivals' total effort level. The prize is awarded to one of the players.

Let $v_i$, for $i = 1, \ldots, n$, represent player $i$'s valuation for the prize. Without loss of generality, we assume that $v_1 \geq v_2 \geq \ldots \geq v_n > 0$. Let $\alpha_i$ represent player $i$'s initial probability of winning the prize, so that $\alpha_i \geq 0$ and $\sum_{j=1}^{n} \alpha_j = 1$. We assume that each player's valuation for the prize and her initial probability of winning the prize are publicly known.

Let $x_i$ represent the effort level expended by player $i$, and let $X$ represent the effort level expended by all the players, so that $X \equiv \sum_{j=1}^{n} x_j$. Let $\mathbf{x}$ denote an $n$-tuple vector of effort levels, one for each player: $\mathbf{x} \equiv (x_1, \ldots, x_n)$. Each player's effort level is nonnegative, and is measured in units commensurate with the prize. Let $p_i$ denote the (final) probability that player $i$ wins the prize. We assume that each player's (final) probability of winning depends on her initial probability of winning and the players' effort levels. More specifically, we assume the following contest success function for player $i$:

$$p_i = \theta \alpha_i + (1 - \theta)f_i(\mathbf{x}),$$

(1)

where $0 < \theta < 1$, $f_i(\mathbf{x}) = x_i/X$ if $X > 0$, and $f_i(\mathbf{x}) = 1/n$ if $X = 0$. This contest success function says that, *ceteris paribus*, player $i$'s probability of winning is increasing in her initial probability of winning; it is increasing in her effort level at a decreasing rate; however, it is decreasing in the rivals' total effort level at a decreasing rate. Function (1) implies that player $i$'s probability of winning may be positive even though she expends zero effort. One interpretation of function (1) is that player $i$'s probability of winning is determined only by her initial probability of winning with probability $\theta$, and is determined only by the value of $f_i(\mathbf{x})$ with probability $(1-\theta)$. Another
interpretation is that player $i$'s probability of winning is determined by a weighted average of her initial probability of winning and the value of $f_i(x)$, where the weights are given by $\theta$ and $(1-\theta)$. We assume that the impact parameter $\theta$ is exogenous and its value is publicly known.

We formally consider the following noncooperative simultaneous-move game. At the beginning of the game, the players know their valuations for the prize, their initial probabilities of winning the prize, and the value of $\theta$. Next, they expend their effort simultaneously and independently in order to win the prize. Finally, the winner is determined.

Let $\pi_i$ represent the expected payoff for player $i$. Then the payoff function for player $i$ is

$$\pi_i = v_i \{ \theta \alpha_i + (1-\theta)f_i(x) \} - x_i.$$ (2)

Recall that, after choosing an effort level of $x_i$, player $i$ earns a payoff of $v_i$ if she wins the prize and 0 if she loses it, and that her probability of winning the prize is $\theta \alpha_i + (1-\theta)f_i(x)$.

We assume that all of the above is common knowledge among the players. We employ Nash equilibrium as the solution concept.

3. Nash equilibrium: Who are the active players?

Obtaining the Nash equilibrium of the game, this section shows that all the players may not be active — that is, all the players may not expend positive effort — in equilibrium. It shows, however, that at least two players are active in equilibrium. Another interesting finding in this section is that the possibility that the winner is determined by the players' initial probabilities of winning reduces prize dissipation, compared to the contest without this possibility. Section 3 also shows implicitly that the winner may not be an active player.

To obtain a Nash equilibrium of the game, we begin by considering the following maximization problem facing player $i$ for $i = 1, \ldots, n$: Maximize player $i$'s expected payoff $\pi_i$ in function (2) over her effort level, $x_i \geq 0$, given effort levels of all the other players. We obtain Lemma 1.6.
Lemma 1. (a) The strategy profile, \( x = (0, \ldots, 0) \), at which every player expends zero effort does not constitute a Nash equilibrium. (b) A strategy profile at which only one player expends positive effort does not constitute a Nash equilibrium.

The proof of Lemma 1 is straightforward. To prove part (a), we use the fact that each player has an incentive to increase her effort level from zero to an infinitesimally small positive, \( \epsilon \), given zero effort levels of the other players. To prove part (b), we use the fact that the player who expends positive effort has an incentive to decrease her effort level, given zero effort levels of the other players.

Let \( x_{-i} \) denote an \((n-1)\)-tuple vector of effort levels, one for each player except player \( i \): 
\[
    x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n). 
\]
Let \( X_{-i} = \sum_{j \neq i} x_j \). Utilizing Lemma 1, to obtain a Nash equilibrium of the game, it suffices to consider the following simplified maximization problem facing player \( i \): Given a vector of effort levels, \( x_{-i} \), such that \( X_{-i} > 0 \),
\[
    \max_{x_i} \pi_i = v_i \{ \theta \alpha_i + (1 - \theta)(x_i/X) \} - x_i \tag{3}
\]
subject to the nonnegativity constraint, \( x_i \geq 0 \).

Let \( x_i^b \) denote player \( i \)'s best response to \( x_{-i} \). Then, by definition, it satisfies the first-order condition:
\[
    \{v_i(1 - \theta)X_{-i}/(X_{-i} + x_i^b)^2\} - 1 = 0 \quad \text{for } x_i^b > 0 \tag{4}
\]

or
\[
    \{v_i(1 - \theta)X_{-i}/(X_{-i} + x_i^b)^2\} - 1 \leq 0 \quad \text{for } x_i^b = 0. \tag{5}
\]

It is straightforward to see that the objective function in expression (3) is strictly concave in \( x_i \), which implies that the second-order condition is satisfied and \( x_i^b \) is unique.
Let $x^*$ denote a Nash equilibrium: $x^* = (x_1^*, \ldots, x_n^*)$. Let $X^* = \sum_{j=1}^{n} x_j^*$. At the Nash equilibrium, player $i$'s effort level is the best response to the other players' effort levels. Thus it satisfies either expression (6) or (7), given the other players' equilibrium effort levels:

$$\{v_i(1 - \theta)(X^* - x_i^*)/(X^*)^2\} - 1 = 0 \quad \text{for } x_i^* > 0 \quad (6)$$

or

$$\{v_i(1 - \theta)(X^* - x_i^*)/(X^*)^2\} - 1 \leq 0 \quad \text{for } x_i^* = 0, \quad (7)$$

which come from expressions (4) through (5), respectively. Using expressions (6) and (7), it is straightforward to obtain Lemma 2, which compares the players' equilibrium effort levels.

**Lemma 2.** At a Nash equilibrium, any two players with the same valuation for the prize expend the same effort. The effort level of a player with a higher valuation, if positive, is greater than that of a player with a lower valuation; otherwise, it is equal to that of a player with a lower valuation. Thus $x_{k-1}^* \geq x_k^*$ holds for $k = 2, \ldots, n$.

Lemma 2 follows from the following facts. The first is that, ceteris paribus, the marginal gross payoff (of increasing effort) of a higher-valuation player is greater than that of a lower-valuation player at any effort level. The second is that each player's marginal gross payoff is decreasing in her own effort level. The third fact is that the marginal cost of increasing effort is constant and the same for all the players. Lemma 2 says that the equilibrium effort level of a highest-valuation player is the largest, that of a lowest-valuation player is the smallest, and the rest are in between.

We are now prepared to obtain the Nash equilibrium of the game. Let $n^*$ denote the number of the players who expend positive effort at the Nash equilibrium. Then it follows from Lemma 1 that $2 \leq n^* \leq n$. It follows from Lemma 2 that players 1 through $n^*$ expend positive
effort, and the rest, if any, expend zero effort. Using expressions (6) and (7), we obtain the Nash equilibrium (see Appendix A).

**Proposition 1.** (a) The following strategy profile constitutes the Nash equilibrium of the game.

Player $i$, for $i = 1, \ldots, n^*$, plays the strategy

$$x_i^* = (1 - \theta)(n^* - 1)\{v_i \sum_{j=1}^{n^*} (1/v_j) - n^* + 1\}/v_i \sum_{j=1}^{n^*} (1/v_j)^2,$$

where $v_n^* \sum_{j=1}^{n^*} (1/v_j) - n^* + 1 > 0$.

Player $k$, for $k = n^* + 1, \ldots, n$, if any, expends zero effort, in which case $v_k \sum_{j=1}^{n^*} (1/v_j) - n^* + 1 \leq 0$ holds. (b) At the Nash equilibrium, the total effort level of the players

is $X^* = (1 - \theta)(n^* - 1)\sum_{j=1}^{n^*} (1/v_j)$.

Under the assumption that $v_1 \geq v_2 \geq \ldots \geq v_n > 0$, part (a) of Proposition 1 implies that $v_i \sum_{j=1}^{n^*} (1/v_j) - n^* + 1 > 0$ and $v_i \sum_{j=1}^{i} (1/v_j) - i + 1 > 0$ hold for $i = 1, \ldots, n^*$. Note that the top two players (according to valuation for the prize) are always active in equilibrium – in other words, players 1 and 2 expend positive effort, regardless of the number of players, their valuations for the prize, their initial probabilities of winning the prize, and the value of $\theta$.

Proposition 1 implies the following results. First, the equilibrium number of active players and their identities are determined only by the players' valuations for the prize. Indeed, they depend neither on the players' initial probabilities of winning the prize nor on the impact parameter $\theta$. This can be explained as follows. Rewriting function (2) as

$$\pi_i = v_i(1 - \theta)x_i + \{v_i(1 - \theta)f_j(x) - x_i\},$$

we see that the equilibrium effort levels of the players are equal to those obtained in a reduced contest in which the same $n$ players compete to win the prize in the absence of initial probabilities of winning and their valuations for the prize are reduced proportionately by a factor of $(1-\theta)$. Accordingly, the equilibrium number of active players and their identities do not depend on the players' initial probabilities of winning. They do not depend on the impact parameter $\theta$, either, because any proportionate decrease in the
players' valuations do not affect the positivity and nonpositivity conditions in Proposition 1, which are used to identify the active or inactive players. Second, the equilibrium effort levels of the players do not depend on the players' initial probabilities of winning the prize. However, they do depend on the impact parameter $\theta$. Indeed, the possibility that the winner is determined by the players' initial probabilities of winning — in other words, any positive value of the impact parameter $\theta$ — reduces each active player's equilibrium effort level and thus the equilibrium total effort level, compared to the contest without this possibility. All these follow from two facts. The first is that the equilibrium effort levels of the players are equal to those obtained in a reduced contest in which the same $n$ players compete to win the prize in the absence of initial probabilities of winning and their valuations for the prize are reduced proportionately by a factor of $(1-\theta)$. The second fact is that any proportionate decrease in the players' valuations decreases the equilibrium effort levels of the active players proportionately. More specifically, a decrease in each active player's valuation decreases her marginal gross payoff at every effort level while her marginal cost remains unchanged, so that her equilibrium effort level decreases. Third, in equilibrium, players who have relatively low valuations for the prize may not be active. Fourth, the equilibrium effort level of each active player, and thus the equilibrium total effort level, depends on the valuations of the active players, but not directly on the valuations of the inactive players. Fifth, any player can be the winner, except the players who have zero initial probabilities of winning and expend zero effort.

If we assume that $v_1 \geq v_j = v$ for $j = 2, \ldots, n$, Proposition 1 is reduced to Corollary 1.

**Corollary 1.** In the case where $v_1 \geq v_j = v$ for $j = 2, \ldots, n$, we have $n^* = n$ at the Nash equilibrium. Player 1's equilibrium effort level is

$$x_1^* = v_1v(1-\theta)(n-1)/\{(n-1)v_1 + v\},$$

and player $j$'s equilibrium effort level is

$$x_j^* = v_1v^2(1-\theta)(n-1)/\{(n-1)v_1 + v\}^2,$$

so that the equilibrium total effort level is

$$X^* = v_1v(1-\theta)(n-1)/\{(n-1)v_1 + v\}.$$
Corollary 1 says that all the players in the contest are active in equilibrium, regardless of
the size of $v_1$, if players 2 through $n$ have the same valuation for the prize. This can be explained
as follows. According to Proposition 1, in equilibrium, player 1’s decision on her effort level
always leaves some room for a second top player to expend positive effort. If players 2 through
$n$ have the same valuation for the prize, then the $(n-1)$ players all are second top players, and
thus expend the same positive effort. An interesting implication of this result is that, facing one
giant rival contestant, the rest never give up, no matter how small each may be, if they are equal
(in terms of valuation for the prize).

If we further assume that $v_i = v$ for all $i = 1, \ldots, n$, then Corollary 1 is reduced to the
following: $x_i^* = v(1 - \theta)(n - 1)/n^2$ and $X^* = v(1 - \theta)(n - 1)/n$. In this case, even though the
players’ initial probabilities of winning differ, the players expend the same effort. Note that the
equilibrium total effort level is less than the players’ common valuation $v$ for the prize—in other
words, underdissipation of the prize occurs.

Next, we look at the expected payoffs for the players at the Nash equilibrium. Let $\pi^*_i$
represent the equilibrium expected payoff for player $i$ for $i = 1, \ldots, n$. Using function (2) and
Proposition 1, we obtain Proposition 2.

**Proposition 2.** At the Nash equilibrium, the expected payoff for player $i$ is equal to

$$\pi^*_i = v_i \theta \alpha_i + (1 - \theta)\{v_i \sum_{j=1}^{n^*} (1/v_j) - n^* + 1\}^2/v_i \{\sum_{j=1}^{n^*} (1/v_j)^2\}$$

for $i = 1, \ldots, n^*$, and

$$\pi^*_i = v_i \theta \alpha_i$$

for $i = n^* + 1, \ldots, n$. In the case where $v_1 \geq v_j = v$ for $j = 2, \ldots, n$, we have

$$\pi^*_1 = v_1 \theta \alpha_1 + v_1 (1 - \theta)\{(n - 1)v_1 - (n - 2)v_1\}^2/\{(n - 1)v_1 + v_1\}^2$$

and

$$\pi^*_j = v \theta \alpha_j + v^3 (1 - \theta)/\{(n - 1)v_1 + v\}^2.$$ 

In the case where $v_i = v$ for all $i = 1, \ldots, n$, we have $\pi^*_i = v\{\theta \alpha_i + (1 - \theta)/n^2\}$. 
Note that the second term in the right-hand side of the first expression is positive for player $i$ for $i = 1, \ldots, n^*$. This says that each active player's net expected payoff from expending effort is positive.

Proposition 2 says that the equilibrium expected payoff for each player depends on her own initial probability of winning the prize, the impact parameter $\theta$, the players' valuations for the prize, and the equilibrium number of active players. Thus it is easy to see that $\pi_{k-1}^* \geq \pi_k^*$ may not hold for $k = 2, \ldots, n$. Next, consider any two players with the same initial probability of winning the prize. Proposition 2 together with Lemma 2 implies that the equilibrium expected payoff for a player with a higher valuation, if positive, is greater than that of a player with a lower valuation. Finally, consider the case where the players have the same valuation for the prize. Proposition 2 implies that the equilibrium expected payoff for a player increases, compared to the contest without initial probabilities of winning the prize, if her initial probability of winning the prize is greater than $1/n^2$. It implies also that a player with a higher initial probability of winning the prize has a greater equilibrium expected payoff than a player with a lower initial probability of winning the prize.

We end this section by discussing contests with only two active players. Contests, with or without initial probabilities of winning, in which there are just two active players are easily observed in the real world. Examples include various types of rent-seeking contests with two active players, election campaigns between two active parties or candidates, and R&D competition between two active firms. An interesting question is then: When do we have just two active players? This paper provides a good answer to that question. We have just two active players in equilibrium if player 3's valuation for the prize is "sufficiently" less than that of player 1 or that of player 2. To be more specific, it follows from Proposition 1 that only players 1 and 2 are active if $v_3 \leq v_1v_2/(v_1 + v_2)$ holds. For example, in the case where $v_1 = v_2$, only players 1 and 2 are active if player 3's valuation is less than or equal to half the valuation of players 1 and 2. Interestingly, we have $x_1^*/v_1 = x_2^*/v_2$ or, equivalently, $x_1^*/x_2^* = v_1/v_2$ at the Nash equilibrium at which only the top two players are active (see Baik 2004).
4. The effects of changing the parameters

We first examine how the outcomes of the game – specifically, the equilibrium number of active players and their identities, the equilibrium effort levels of the players, and the equilibrium expected payoffs for the players – respond when the players' initial probabilities of winning change. Proposition 3 is immediate from Propositions 1 and 2.

**Proposition 3.** If the players' initial probabilities of winning the prize change, then (a) the equilibrium number $n^*$ of active players and their identities remain unchanged, (b) the equilibrium effort level of each player, and thus the equilibrium total effort level, remains unchanged, and (c) the equilibrium expected payoff for each player changes in the same direction as her initial probability of winning the prize does.

Proposition 3 says that changes in the players' initial probabilities of winning do not affect the equilibrium number of active players, their identities, or the equilibrium effort levels of the players, but do affect the equilibrium expected payoffs for the players.

Next, we examine the effects of increasing the impact parameter $\theta$, *ceteris paribus*, on the outcomes of the game. Using Propositions 1 and 2, we obtain Proposition 4.

**Proposition 4.** As the impact parameter $\theta$ increases, (a) the equilibrium number of active players and their identities remain unchanged, (b) the equilibrium effort level of each active player, and thus the equilibrium total effort level, decreases, and (c) the equilibrium expected payoff for an inactive player, if any, whose initial probability of winning the prize is positive, increases.

If the impact parameter $\theta$ increases, then, according to function (1), player $i$'s probability of winning is determined to be such that more weight is given to her initial probability of winning and less weight to the value of $f_i(x)$. Consequently, each active player is less motivated
to expend effort, and actually expends less effort than before the change. The following complements part (c) of Proposition 4. As the impact parameter \( \theta \) increases, the equilibrium expected payoff for an active player increases, decreases, or remains unchanged, depending on the relative magnitudes of the two expressions, \( \nu_i \alpha_i \) and \( \left\{ \nu_i \sum_{j=1}^{n^*} (1/\nu_j) - n^* + 1 \right\}^2 / \nu_i \left\{ \sum_{j=1}^{n^*} (1/\nu_j) \right\}^2. \)

Finally, we examine the effects of changing the players' valuations for the prize on the outcomes of the game.\(^{11}\) Using Propositions 1 and 2, we obtain Proposition 5.

**Proposition 5.** If the valuations of one or more active players increase, then the equilibrium number of active players never increases, but may decrease. If the valuations of one or more inactive players decrease, then (a) the equilibrium number of active players and their identities remain unchanged, (b) the equilibrium effort level of each player, and thus the equilibrium total effort level, remains unchanged, and (c) their equilibrium expected payoffs, if positive, decrease while the equilibrium expected payoffs of the rest of the players remain unchanged.

The proof of Proposition 5 is straightforward, and therefore omitted. In addition to Proposition 5, we obtain the following results. First, if the valuations of the active players increase at the same rate, then the equilibrium number of active players and their identities remain unchanged; however, their equilibrium effort levels and their equilibrium expected payoffs, respectively, increase at the same rate, and furthermore increase at the very rate that their valuations increase. Second, if the valuations of all the players increase (decrease) proportionately, then the equilibrium number of active players and their identities remain unchanged; however, the equilibrium effort levels of the active players and the equilibrium expected payoffs (if positive) of the players, respectively, increase (decrease) proportionately.
5. The model with the all-pay-auction selection rule

In this section, we present a model which is the same as the model in Section 2 with the exception that \( f_i(x) \) in function (1) is now replaced with

\[
h_i(x) = \begin{cases} 
1/m & \text{if player } i \text{ is one of } m \text{ players expending the largest effort} \\
0 & \text{if } x_i < x_k \text{ holds for some } k,
\end{cases}
\]

where \( 1 \leq m \leq n \) and \( k \neq i \). Thus, in the present model, the contest success function for player \( i \) is given by

\[
p_i = \theta \alpha_i + (1 - \theta)h_i(x),
\]

and the payoff function for player \( i \) is given by

\[
\pi_i = v_i \{ \theta \alpha_i + (1 - \theta)h_i(x) \} - x_i.
\]

Function (8) implies that, unlike in standard all-pay auctions, any of the players who expend the largest effort (or bid the highest bid) may not win the prize due to the players' initial probabilities of winning the prize. It implies also that player \( i \)'s effort only affects her final probability of winning if her effort level \( x_i \) is the largest one.

As in the model in Section 2, we formally consider the following noncooperative simultaneous-move game. At the beginning of the game, the players know their valuations for the prize, their initial probabilities of winning the prize, and the value of \( \theta \). Next, they expend their effort simultaneously and independently in order to win the prize. Finally, the winner is determined.

We assume that all of the above is common knowledge among the players. We employ Nash equilibrium as the solution concept.
5.1. *Nash equilibria in mixed strategies*

We first prove the nonexistence of a pure-strategy Nash equilibrium of the game, and then obtain its mixed-strategy Nash equilibria.

**Proposition 6.** There is no Nash equilibrium in pure strategies.

*Proof.* Consider a strategy profile of \((x_1, \ldots, x_n)\). First, consider the case where there is only one player, say player \(t\), expending the largest effort. In this case, given the other players' strategies, player \(t\) has an incentive to deviate from her strategy. Specifically, she can increase her expected payoff by reducing her effort level to \((x_t - \epsilon)\) where \(\epsilon\) is an infinitesimally small positive. Next, consider the case where there are \(m\) players expending the largest effort, and each such player's effort level is less than her valuation times \((1 - \theta)\), where \(1 < m \leq n\). In this case, given the other players' strategies, each of the \(m\) players has an incentive to deviate from her strategy. Specifically, she can increase her expected payoff by slightly increasing her effort level. Finally, consider the case where there are \(m\) players expending the largest effort, and some such player's effort level, say player \(k\)'s effort level, is greater than or equal to \(v_k(1 - \theta)\), where \(1 < m \leq n\). In this case, given the other players' strategies, player \(k\) has an incentive to deviate from her strategy. Specifically, she can increase her expected payoff by reducing her effort level to zero. Therefore, there is no Nash equilibrium in pure strategies.

Next, we obtain the mixed-strategy Nash equilibria of the game. We do so considering three different cases with respect to the players' valuations for the prize: (i) the case where \(v_1 \geq v_2 > v_3 \geq \cdots \geq v_n\), (ii) the case where \(v_1 > v_2 = \cdots = v_s > v_{s+1} \geq \cdots \geq v_n\) for \(3 \leq s \leq n\), and (iii) the case where \(v_1 = v_2 = \cdots = v_t > v_{t+1} \geq \cdots \geq v_n\) for \(3 \leq t \leq n\). In the first case, there is a unique mixed-strategy Nash equilibrium, which is reported in Proposition 7; however, in the last two cases, there are multiple mixed-strategy Nash equilibria (see Appendix B). Let \(G^{**}_i(x_i)\) represent a cumulative distribution function of \(x_i\) – that is, a mixed strategy of player
Proposition 7. (a) If \( v_1 \geq v_2 > v_3 \geq \cdots \geq v_n \), then there is a unique mixed-strategy Nash equilibrium at which 
\[
G_1^{**}(x_1) = \left[ x_1 / v_2(1 - \theta) \right] I_{[0, v_2(1 - \theta)]}(x_1) + I_{(v_2(1 - \theta), \infty)}(x_1), \\
G_2^{**}(x_2) = \left[ \{ x_2 / v_1(1 - \theta) \} + (v_1 - v_2) / v_1 \right] I_{[0, v_1(1 - \theta)]}(x_2) + I_{(v_1(1 - \theta), \infty)}(x_2), \\
G_k^{**}(x_k) = I_{(0, \infty)}(x_k) \text{ for } k = 3, \ldots, n, 
\]
where \( I(Z) \) is the indicator function of a set \( Z \).

(b) At the mixed-strategy Nash equilibrium, we have 
\[
Ex_1^{**} = v_2(1 - \theta)/2, \\
Ex_2^{**} = v_2^2(1 - \theta)/2v_1, \text{ } Ex_k^{**} = 0 \text{ for } k = 3, \ldots, n, \text{ and } EX^{**} = v_2(1 - \theta)(v_1 + v_2)/2v_1. 
\]

The proof of part (a) is similar to the proofs contained in Hillman and Riley (1989) and Baye et al. (1996), and that of part (b) is straightforward, so that the proof of Proposition 7 is omitted. Note that only players 1 and 2, the top two players according to valuation for the prize, are active at the mixed-strategy Nash equilibrium – specifically, players 1 and 2 play the mixed strategies, \( G_1^{**}(x_1) \) and \( G_2^{**}(x_2) \), respectively, and all the other players use the pure strategy of 0. Note also that \( G_1^{**}(x_1) \) and \( G_2^{**}(x_2) \) have the same support of \([0, v_2(1 - \theta)]\).

Proposition 7 implies the following results, which are similar to those implied by Proposition 1. First, the equilibrium number of active players and their identities are determined only by the players' valuations for the prize. Indeed, they depend neither on the players' initial probabilities of winning the prize nor on the impact parameter \( \theta \). This can be explained as follows. Rewriting function (9) as 
\[
\pi_i = v_i(1 - \theta)h_i(x) - x_i, 
\]
we see that the equilibrium expected effort levels of the player are equal to those obtained in a reduced contest in which the same \( n \) players compete to win the prize in the absence of initial probabilities of winning and their valuations for the prize are reduced proportionately by a factor of \((1 - \theta)\). Accordingly, the equilibrium number of active players and their identities do not depend on the players' initial probabilities of winning. They do not depend on the impact parameter \( \theta \), either, because any proportionate decrease in the players' valuations do not affect the identities of the players.\(^{13}\)
top two players. Second, the equilibrium expected effort levels of the players do not depend on the players’ initial probabilities of winning the prize. However, they do depend on the impact parameter $\theta$. Indeed, the possibility that the winner is determined by the players’ initial probabilities of winning—in other words, any positive value of the impact parameter $\theta$—reduces the equilibrium expected effort levels of the two active players, and thus the equilibrium expected total effort level, compared to the contest without this possibility. All these follow from two facts. The first is that the equilibrium expected effort levels of the players are equal to those obtained in a reduced contest in which the same $n$ players compete to win the prize in the absence of initial probabilities of winning and their valuations for the prize are reduced proportionately by a factor of $(1-\theta)$. The second fact is that any proportionate decrease in the players’ valuations decreases the equilibrium expected effort levels of the active players proportionately. Third, we have $E_{x_1}^{**} \geq E_{x_2}^{**} > E_{x_3}^{**} = \cdots = E_{x_n}^{**} = 0$. Fourth, the equilibrium expected effort levels, $E_{x_1}^{**}$ and $E_{x_2}^{**}$, of the two active players, and thus the equilibrium expected total effort level, $E_{X}^{**}$, depend only on the valuations of the two active players. Fifth, player $k$, for $k = 3, \ldots, n$, can never be the winner if her initial probability of winning the prize is zero.

Next, we look at the expected payoffs for the players at the mixed-strategy Nash equilibrium in the case where $v_1 \geq v_2 > v_3 \geq \cdots \geq v_n$. Let $\pi_i^{**}$ represent the equilibrium expected payoff for player $i$ for $i = 1, \ldots, n$. Using function (9) and Proposition 7, we obtain Proposition 8.

**Proposition 8.** Consider the case where $v_1 \geq v_2 > v_3 \geq \cdots \geq v_n$. At the mixed-strategy Nash equilibrium, if $v_1 = v_2$, the expected payoff for player $i$ is equal to $\pi_i^{**} = \theta \alpha_i$ for $i = 1, \ldots, n$. In the case where $v_1 > v_2$, we have $\pi_1^{**} = \theta \alpha_1 + (1-\theta)(v_1 - v_2)$ and $\pi_k^{**} = \theta \alpha_k$ for $k = 2, \ldots, n$.

Proposition 8 says that the equilibrium expected payoff for each player, except player 1 in the case where $v_1 > v_2$, depends only on her own initial probability of winning the prize and
the impact parameter $\theta$. Indeed, it does not depend on the players' valuations for the prize. Thus it is easy to see that a player with a higher initial probability of winning the prize has a greater equilibrium expected payoff than a player with a lower initial probability of winning the prize.\textsuperscript{18} It is also easy to see that the equilibrium expected payoff for a player with a higher valuation may be less than that of a player with a lower valuation. Another interesting observation from Proposition 8 is that the equilibrium expected payoff for each player (except player 1 in the case where $v_1 > v_2$) increases, compared to the contest without initial probabilities of winning the prize, if her initial probability of winning the prize is positive (see footnote 15). On the other hand, in the case where $v_1 > v_2$, the equilibrium expected payoff for player 1 decreases, compared to the contest without initial probabilities of winning the prize, if $\alpha_1 < (v_1 - v_2)$ \textsuperscript{2} holds. Based on these, we conclude that the possibility that the winner is determined by the players' initial probabilities of winning — in other words, any positive value of the impact parameter $\theta$ — may make player 1 worse off but makes better off other players with positive initial probability of winning, compared to the contest without this possibility.

5.2. \textit{The effects of changing the parameters}

As in Section 4, we examine the effects of changing the players' initial probabilities of winning the prize on the outcomes of the game — specifically, the equilibrium number of active players and their identities, the equilibrium expected effort levels of the players, and the equilibrium expected payoffs for the players. We examine also the effects of increasing the impact parameter $\theta$ and the effects of changing the players' valuations for the prize. In the case where $v_1 \geq v_2 > v_3 \geq \cdots \geq v_n$, we obtain almost the same results as in Propositions 3, 4, and 5.\textsuperscript{19}

6. \textbf{An extension: Endogenizing the impact parameter $\theta$}

In the models we have considered so far, the impact parameter $\theta$ is exogenous. In this section, we consider an extended model in which it is endogenous. The extended model is the
same as the model in Section 2 with the exception that, in the extended model, the impact parameter \( \theta \) is a nonincreasing function of the total effort level \( X \) of the players. Interestingly, we show that every player may expend zero effort in equilibrium.

Let the impact parameter \( \theta \) be a function of the total effort level \( X \) of the players: \( \theta = \theta(X) \). We assume that the function \( \theta \) has the following properties.

**Assumption 1.** Let \( \theta \) be a function from \( R_+ \) to the unit interval \([0, 1]\), where \( R_+ \) denotes the set of all positive real numbers. We assume that \( \theta(0) = 1; \theta(X) = 0 \) for all \( X \) with \( X \geq M \), where \( M \in (0, \infty] \); and \( \lim_{X \to \infty} \theta(X) = 0 \). We assume also that the function \( \theta \) is continuous on \( R_+ \) and \( \theta'(X) < 0 \) for all \( X \) with \( 0 \leq X < M \), where \( \theta' \) denotes the first derivative of the function \( \theta \).

In Assumption 1, we assume that each player's probability of winning is determined only by her initial probability of winning if the total effort level \( X \) of the players is equal to zero. We assume also that, as the total effort level \( X \) of the players increases, the impact parameter \( \theta \) decreases, which implies that player \( i \)'s probability of winning is determined to be such that less weight is given to her initial probability of winning and more weight to the value of \( f_i(x) \). Assumption 1 may reflect the idea that as the total effort level \( X \) of the players increases, the decision-maker who has authority to select the winner should value the players' effort more and give less weight to their initial probabilities of winning.

One example of the function \( \theta \) which satisfies the properties in Assumption 1 is
\[
\theta(X) = 1 - \min\{\delta X, v_i/V, \}
\]
where \( \delta > 0 \) and \( V \equiv \sum_{j=1}^{n} v_j \). Another example of the function \( \theta \) is
\[
\theta(X) = \exp(-\eta X), \quad \text{where} \quad \eta > 0.
\]
Note that \( M = V/\delta \) in the first example, and that \( M = \infty \) in the second example.

The contest success function for player \( i \) is then
\[
p_i = \theta(X)\alpha_i + (1 - \theta(X))f_i(x),
\]
and her payoff function is
\[ \pi_i = v_i \{ \theta(X) x_i + (1 - \theta(X)) f_i(x) \} - x_i. \]  

We formally consider the following game. At the start of the game, the players know their valuations for the prize, their initial probabilities of winning the prize, and the function \( \theta \). Next, they expend their effort simultaneously and independently in order to win the prize. Finally, the winner is determined.

Now we show that, under certain conditions, a Nash equilibrium of the game exists in which every player expends zero effort. We let \( x^N \) denote the Nash equilibrium of a game which is the same as the game in Section 2 or Section 6 with the exception that the players compete to win the prize in the absence of initial probabilities of winning, where \( x^N \equiv (x_1^N, \ldots, x_n^N) \). We let \( X^N \equiv \sum_{j=1}^{n} x_j^N \).

**Proposition 9.** (a) If \( \theta'(X)(\alpha_i - 1) \leq 1/v_i \) for all \( i = 1, \ldots, n \) and for all \( X \) with \( 0 \leq X < M \), then the strategy profile, \( x = (0, \ldots, 0) \), at which every player expends zero effort constitutes a Nash equilibrium. In this case, there may exist multiple pure-strategy Nash equilibria. (b) If \( \theta'(X)(\alpha_i - x_i/X) + (1 - \theta(X))(X - x_i)/X^2 < 1/v_i \) for all \( i = 1, \ldots, n \) and for all \( X \) with \( 0 \leq X < M \), and if \( X^N \leq M \), then the strategy profile, \( x = (0, \ldots, 0) \), at which every player expends zero effort is a unique Nash equilibrium.

The proof of Proposition 9 is provided in Appendix C. Proposition 9 says that every player may expend zero effort in equilibrium. This result contrasts with Lemma 1, and further with the received result – that the no-effort equilibrium does not exist – in the literature on the theory of contests. Proposition 9 says also that there may exist multiple pure-strategy Nash equilibria, which contrasts with the well-known result – that the Nash equilibrium is unique in contests with the simplest logit-form functions – in the literature on the theory of contests.
The existence of the no-effort equilibrium appears counterintuitive because one may expect that player $i$ is more motivated to expend effort if an increase in $X$ gives more weight to the value of $f_i(x)$. However, this appearance is wrong. One should not overlook the fact that, when increasing her effort level, player $i$ not only increases her cost but also decreases the weight to be given to her initial probability of winning. Indeed, the no-effort equilibrium occurs if the specified conditions are met, because player $i$'s marginal gross payoff at every effort level does not exceed her marginal cost given zero effort levels of the other players.

Note that the specified conditions are more likely to hold — thus the no-effort equilibrium is more likely to occur — as the players' valuations for the prize decrease.

To better understand Proposition 9, consider a contest in which $n = 2$, $v_1 = v_2 = 1/2$, $\alpha_1 = \alpha_2 = 1/2$, and $\theta(X) = 1 - \min\{\delta X, V\}/V$, where $\delta > 0$ and $V \equiv v_1 + v_2$. In this contest, if $\delta = 4$, then the strategy profile, $x = (0, 0)$, at which both players expend zero effort is not the only Nash equilibrium; indeed, the strategy profiles at which one player expends zero effort and the other player expends an effort level of $1/8$ also are Nash equilibria. If $0 < \delta < 4$, then the strategy profile, $x = (0, 0)$, is a unique Nash equilibrium. Next, considering a contest in which $n = 2$, $v_1 = v_2$, $\alpha_1 = \alpha_2$, and $\theta(X) = \exp(-X)$, we find that the strategy profile, $x = (0, 0)$, is a unique Nash equilibrium.

7. Conclusions

We have studied contests in which $n$ risk-neutral players each want to win a prize, each player has an initial probability of winning the prize, and the players compete with one another by expending irreversible effort to win the prize. We have formally considered the following simultaneous-move game. At the beginning of the game, the players know their valuations for the prize, their initial probabilities of winning the prize, and the information regarding the value of $\theta$. Next, they expend their effort simultaneously and independently in order to win the prize. Finally, the prize is awarded to one of the players.
In Section 2, we have considered the model with the simplest logit-form function in which the impact parameter $\theta$ is exogenous. In Section 3, obtaining the Nash equilibrium of the game, we have shown the following. First, all the players may not be active in equilibrium, but at least two players are active. Second, the equilibrium number of active players and their identities depend neither on the players' initial probabilities of winning the prize nor on the impact parameter $\theta$. Third, the equilibrium effort levels of the players do not depend on the players' initial probabilities of winning the prize. However, the possibility that the winner is determined by the players' initial probabilities of winning reduces each active player's equilibrium effort level, compared to the contest without this possibility. Fourth, the winner may not be an active player. Fifth, in the case where the players have the same valuation for the prize, the equilibrium expected payoff for a player increases, compared to the contest without initial probabilities of winning the prize, if her initial probability of winning the prize is greater than $1/n^2$. In Section 4, we have examined how these outcomes of the game respond when the players' initial probabilities of winning the prize, the impact parameter $\theta$, or the players' valuations for the prize change.

In Section 5, we have considered the model with the all-pay-auction selection rule in which the impact parameter $\theta$ is exogenous. Focusing on the case where $v_1 \geq v_2 > v_3 \geq \cdots \geq v_n$, we have shown the following. First, only the top two players according to valuation for the prize are active at the unique mixed-strategy Nash equilibrium. Second, the equilibrium number of active players and their identities depend neither on the players' initial probabilities of winning the prize nor on the impact parameter $\theta$. Third, the equilibrium effort levels of the players do not depend on the players' initial probabilities of winning the prize. However, the possibility that the winner is determined by the players' initial probabilities of winning reduces the equilibrium expected effort levels of the two active players, compared to the contest without this possibility. Fourth, except the top two players, any player can never be the winner if her initial probability of winning the prize is zero. Fifth, a player with a higher initial probability of winning the prize has a greater equilibrium expected payoff than a
player with a lower initial probability of winning the prize. Sixth, the equilibrium expected
payoff for a player with a higher valuation may be less than that of a player with a lower
valuation. Seventh, the possibility that the winner is determined by the players' initial
probabilities of winning may make the highest-valuation player worse off but makes better off
other players with positive initial probability of winning, compared to the contest without this
possibility.

In Section 6, we have considered the extended model in which the impact parameter \( \theta \) is
endogenous, and is a nonincreasing function of the total effort level \( X \) of the players. We have
shown that every player may expend zero effort in equilibrium, and that there may exist multiple
pure-strategy Nash equilibria.

In the models that we have studied in this paper, the players' initial probabilities of
winning the prize are exogenous. It would be interesting to study an extended model in which
the players' initial probabilities of winning the prize are endogenous. In the models that we have
studied in this paper, the winner is selected among the players whose final probabilities of
winning the prize are positive. It would be interesting to study a model in which the winner is
selected only among the active players. Finally, it would be interesting to study a model in
which the players have incomplete information about the value of the impact parameter \( \theta \). We
leave these extensions and modifications for future research.
Footnotes

1. A contest is defined as a situation in which players or contestants compete with one another by expending irreversible effort or resources to win a prize. Examples of contests include rent-seeking contests, R&D contests, sporting contests, election campaigns, litigation, tournaments, all-pay auctions, environmental conflicts, and arms races. Admittedly, due to their prevalence and importance, such contests have been and are studied by many economists. Important work in the literature on the theory of contests includes Loury (1979), Tullock (1980), Rosen (1986), Dixit (1987), Hillman and Riley (1989), Hirshleifer (1989), Katz et al. (1990), Ellingsen (1991), Nitzan (1991), Baik and Shogren (1992), Baye et al. (1993), Leininger (1993), Skaperdas (1996), Clark and Riis (1998), Hurley and Shogren (1998), Moldovanu and Sela (2001), Hvide (2002), Che and Gale (2003), Szymanski (2003), Corchon (2007), Epstein and Nitzan (2007), Congleton et al. (2008), Konrad (2009), Siegel (2010), and Flamand and Troumpounis (2015).

2. Note that each player's initial probability of winning can be interpreted as her initial fractional share of the prize or initial fractional claim on the prize.

3. By prize dissipation, we mean the equilibrium total effort level, which is "dissipated" in pursuit of the prize. In the literature on the theory of contests, one of the main issues is how much prize dissipation takes place. This issue is of great importance because the players' efforts are, for example, the firms' R&D expenditures in an R&D contest – which determine the expected date of invention – and interpreted as social costs in a rent-seeking contest.

4. One may interpret \( \alpha_i \) as player i's initial fractional share of the prize or her initial fractional claim on the prize.

5. Unlike the specification in function (1), as player i's contest success function, the simplest logit-form function \( f_i \) by itself has been extensively used in the literature on the theory of contests. Examples include Tullock (1980), Hillman and Riley (1989), Hirshleifer (1989),

6. Lemma 1 still holds true for general contest success functions. For example, it holds true for \( p_i = \theta \alpha_i + (1 - \theta)g_i(x) \), for \( i = 1, \ldots, n \), such that \( g_i(x) \geq 0 \) and \( \sum_{j=1}^{n} g_j(x) = 1 \) for all \( x \in R^n_+ \), \( g_i(0, \ldots, 0) < 1 \), and \( g_i(0, \ldots, 0, x_i, 0, \ldots, 0) = 1 \) for all \( x_i > 0 \).

7. Using these positivity and nonpositivity conditions, we identify the active players at the Nash equilibrium. Proposition 5 in Hillman and Riley (1989) states similar conditions.

8. It follows from part (a) of Proposition 1 that \( \sum_{j=1}^{n^* - 1} (1/v_j) - (n^* - 1) + 1 > 0 \) holds. Then, under the assumption that \( v_1 \geq v_2 \geq \ldots \geq v_n > 0 \), we have \( \sum_{j=1}^{n^* - 1} (1/v_j) - (n^* - 1) + 1 > 0 \), which in turn implies that \( \sum_{j=1}^{n^* - 2} (1/v_j) - (n^* - 2) + 1 > 0 \) holds. Then, under the assumption that \( v_1 \geq v_2 \geq \ldots \geq v_n > 0 \), we have \( \sum_{j=1}^{n^* - 2} (1/v_j) - (n^* - 2) + 1 > 0 \), and so on. Hence, \( \sum_{j=1}^{i} (1/v_j) - i + 1 > 0 \) holds for \( i = 1, \ldots, n^* \).

9. We can obtain the outcomes of the game without initial probabilities of winning by setting \( \theta \) equal to zero in the outcomes of the present game.

10. Studying contests with the selection rule of all-pay auctions, Hillman and Riley (1989) and Baye et al. (1996) show that, if \( v_3 < v_2 \leq v_1 \) holds, then there exists a unique mixed-strategy Nash equilibrium at which only the top two players are active. See also Proposition 7 below.

11. Considering two-player asymmetric contests with ratio-form contest success functions, Baik (2004) examines the effects of changing the players' valuations for the prize on the equilibrium effort ratio, the prize dissipation ratios, and the players' equilibrium effort levels.

12. The function \( h_i \) is the selection rule of all-pay auctions. It by itself has been extensively used as player \( i \)'s contest success function in the literature on the theory of contests. Examples include Hillman and Riley (1989), Baye et al. (1993, 1996), Clark and Riis (1998), Baik et al. (2001), Moldovanu and Sela (2001), Che and Gale (2003), and Barbieri et al. (2014).
13. In terms of the symbols, we have $EX^{**} \equiv \sum_{j=1}^{n} Ex_j^{**} \equiv \sum_{j=1}^{n} \int_{-\infty}^{x_j} dG_j^{**}(x_j)$.

14. Consider contests in which the players' effort or outlays are revenues collected by the contest organizer or bribes given to the decision-maker who has authority to select the winner. In such contests, the players' mixed strategies may be viewed as "lotteries with monetary payoffs" facing the contest organizer or the decision-maker. Then we may say that $G_1^{**}(\cdot)$ first-order stochastically dominates $G_2^{**}(\cdot)$, from the viewpoint of the contest organizer or the decision-maker, because $G_1^{**}(x) \leq G_2^{**}(x)$ for all $x \in R^+$. For the technical term of first-order stochastic dominance, see Mas-Colell et al. (1995, p. 194-97).

15. We can obtain the outcomes of the game without initial probabilities of winning by setting $\theta$ equal to zero in the outcomes of the present game.

16. Thus, unlike in contests without initial probabilities of winning, prize dissipation is always expectationally less than complete (see, for example, Hillman and Riley 1989; Baye et al. 2005).

17. We report in Appendix B the equilibrium expected payoffs for the players for the other two cases.

18. More precisely, in the case where $v_1 = v_2$, we have: $\pi_h^{**} > \pi_k^{**}$ if and only if $\alpha_h > \alpha_k$, for $h, k = 1, \ldots, n$. However, in the case where $v_1 > v_2$, we have: $\pi_h^{**} > \pi_k^{**}$ if and only if $\alpha_h > \alpha_k$, for $h, k = 2, \ldots, n$.

19. Note, however, that the equilibrium expected payoff for every player remains unchanged as the valuations of one or more inactive players decrease.

20. We obtain that $X^N = (n^N - 1) / \sum_{j=1}^{n^N} (1/v_j)$, where $n^N$ denotes the number of the players who expend positive effort at the Nash equilibrium $x^N$ of the game.
Appendix A: Obtaining the equilibrium effort levels of the \(n^*\) active players

From equation (6), we have the following system of \(n^*\) simultaneous equations:

\[
\begin{align*}
  v_1(1 - \theta)(X^* - x_1^*)/(X^*)^2 - 1 &= 0 \\
  v_2(1 - \theta)(X^* - x_2^*)/(X^*)^2 - 1 &= 0 \\
  \vdots &= \vdots \\
  v_{n^*}(1 - \theta)(X^* - x_{n^*}^*)/(X^*)^2 - 1 &= 0.
\end{align*}
\]

These equations can be rewritten as

\[
\begin{align*}
  (X^*)^2/v_1 &= (1 - \theta)(X^* - x_1^*) \\
  (X^*)^2/v_2 &= (1 - \theta)(X^* - x_2^*) \\
  \vdots &= \vdots \\
  (X^*)^2/v_{n^*} &= (1 - \theta)(X^* - x_{n^*}^*). \quad (A1)
\end{align*}
\]

Adding these equations together, we have

\[
\{\sum_{j=1}^{n^*}(1/v_j)\}(X^*)^2 = (1 - \theta)(n^*X^* - X^*).
\]

This yields

\[
X^* = (1 - \theta)(n^* - 1)\sum_{j=1}^{n^*}(1/v_j).
\]

Substituting this expression for \(X^*\) into the equations in (A1), we obtain the equilibrium effort levels of the \(n^*\) active players.
Appendix B: Characterization of the mixed-strategy Nash equilibria of the game for the last two cases

The proof of Proposition B1 below is similar to the proofs contained in Baye et al. (1996), and therefore omitted.

**Proposition B1.** (a) If \( v_1 > v_2 = \cdots = v_s > v_{s+1} \geq \cdots \geq v_n \) for \( 3 \leq s \leq n \), then there are multiple mixed-strategy Nash equilibria at which player 1 plays a mixed strategy, at least one and at most \( s-1 \) players among players 2 through \( s \) play mixed strategies, and all the other players use the pure strategy of 0. At all the mixed-strategy Nash equilibria, the expected payoff for player 1 is \( \pi_1^{**} = \theta \alpha_1 + (1 - \theta)(v_1 - v_2) \), and that for player \( k \) is \( \pi_k^{**} = \theta \alpha_k \) for \( k = 2, \ldots, n \).

(b) If \( v_1 = v_2 = \cdots = v_t > v_{t+1} \geq \cdots \geq v_n \) for \( 3 \leq t \leq n \), then there are multiple mixed-strategy Nash equilibria at which at least two and at most \( t \) players among players 1 through \( t \) play mixed strategies and all the other players use the pure strategy of 0. At all the mixed-strategy Nash equilibria, the expected total effort level of the players is \( EX^{**} = v_1(1 - \theta) \), and the expected payoff for player \( i \) is \( \pi_i^{**} = \theta \alpha_i \) for \( i = 1, \ldots, n \).

Part (a) says that the equilibrium number of active players is at least two and at most \( s \). Part (b) says that the equilibrium number of active players is at least two and at most \( t \), and that player 1, a highest-valuation player, may not be active in equilibrium. It follows from part (b) and Proposition 7 that, if at least two players have the highest valuation for the prize, then the equilibrium expected total effort level \( EX^{**} \) of the players is the same, regardless of the number of players, their initial probabilities of winning the prize, and the valuations for the prize of the other players.
Appendix C: Proof of Proposition 9

(a) Consider the following maximization problem facing player $i$ for $i = 1, ... , n$: Given zero effort levels of the other players,

$$\max_{x_i} \pi_i = v_i\{\theta(X)\alpha_i + (1 - \theta(X))f_i(x)\} - x_i$$

subject to the nonnegativity constraint, $x_i \geq 0$.

Under the given condition that $\theta'(X)(\alpha_i - 1) \leq 1/v_i$ for all $X$ with $0 \leq X < M$, we obtain that $\partial \pi_i/\partial x_i \leq 0$ for all $x_i$ with $0 \leq x_i < M$, and that $\partial \pi_i/\partial x_i = -1$ for all $x_i$ with $x_i \geq M$. This implies that player $i$ has no incentive to deviate from her strategy of expending zero effort, given zero effort levels of the other players.

For the proof of the second claim that there may exist multiple pure-strategy Nash equilibria, we provide an example in the last paragraph of Section 6.

(b) Consider the following maximization problem facing player $i$ for $i = 1, ... , n$: Given zero effort levels of the other players,

$$\max_{x_i} \pi_i = v_i\{\theta(X)\alpha_i + (1 - \theta(X))f_i(x)\} - x_i$$

subject to the nonnegativity constraint, $x_i \geq 0$.

Given zero effort levels of the other players, the first specified condition in part (b) is equivalent to $\theta'(X)(\alpha_i - 1) < 1/v_i$ for all $X$ with $0 \leq X < M$. Under this condition, we obtain that $\partial \pi_i/\partial x_i < 0$ for all $x_i$ with $0 \leq x_i < M$, and that $\partial \pi_i/\partial x_i = -1$ for all $x_i$ with $x_i \geq M$. This implies that, given zero effort levels of the other players, player $i$'s best response is to expend zero effort and is unique. (Note that this in turn implies that there is no Nash equilibrium at which only one player expends positive effort.)

Next, we show that there is no Nash equilibrium at which at least two players expend positive effort. Suppose on the contrary that there is such a Nash equilibrium, denoted by $x^D$,
where \( x^D \equiv (x_1^D, \ldots, x_n^D) \). Let \( N^D \) denote the set of the players who expend positive effort at that Nash equilibrium. Now, consider the following maximization problem facing player \( k \) for \( k \in N^D \): Given the \((n-1)\)-tuple vector \( x_{-k}^D \) of effort levels of the other players,

\[
\begin{align*}
\text{Max} \quad & \pi_k = v_k [\theta(\sum_{j \neq k} x_j^D + x_k) \alpha_k + \{1 - \theta(\sum_{j \neq k} x_j^D + x_k)\} \{x_k / (\sum_{j \neq k} x_j^D + x_k)\}] - x_k \\
\text{subject to the nonnegativity constraint, } x_k & \geq 0.
\end{align*}
\]

Let \( x_k^B \) denote player \( k \)'s best response to \( x_{-k}^D \).

We have two cases to consider: (i) \( \sum_{j \neq k} x_j^D + x_k^B \leq M \) for some \( k \) with \( k \in N^D \) and (ii) \( \sum_{j \neq k} x_j^D + x_k^B > M \) for all \( k \) with \( k \in N^D \). We show that a contradiction arises in each of those two cases.

First, we consider case (i). Under the first specified condition in part \((b)\), we obtain that \( \partial \pi_k / \partial x_k < 0 \) for all \( x_k \) with \( 0 \leq x_k < M - \sum_{j \neq k} x_j^D \). We obtain also that \( \partial \pi_k / \partial x_k \) (or, precisely, the left-side derivative of \( \pi_k \)) is nonpositive at \( x_k = M - \sum_{j \neq k} x_j^D \). This both implies that \( x_k^B \) is equal to 0 and is unique, which contradicts that player \( k \) expends positive effort, \( x_k^D \), at the Nash equilibrium \( x^D \).

Second, we consider case (ii). Since \( \theta(X) = 0 \) for all \( X \) with \( X > M \) due to Assumption 1, we have

\[
\partial \pi_k / \partial x_k = v_k \sum_{j \neq k} x_j^D / (\sum_{j \neq k} x_j^D + x_k)^2 \quad \text{for all } x_k > 0.
\]

If \( x_k^B > 0 \), then, by definition, \( x_k^B \) satisfies the following first-order condition:

\[
v_k \sum_{j \neq k} x_j^D / (\sum_{j \neq k} x_j^D + x_k^B)^2 = 1. \tag{C1}
\]
Since \( x^D \) is a Nash equilibrium, it must satisfy these first-order conditions for all \( k \in N^D \), and satisfy the condition for case (ii) that of \( X^D > M \), where \( X^D \equiv \sum_{j=1}^{n} x^D_j \).

Now, consider a game which is the same as the game in Section 2 or Section 6 with the exception that the players compete to win the prize in the absence of initial probabilities of winning. It is well known in the literature on the theory of contests that the Nash equilibrium of the game is unique: See, for example, Cornes and Hartley (2005) and Yamazaki (2008). The Nash equilibrium \( x^N \) of the game must satisfy the first-order conditions – as in (C1) – for the players who expend positive effort.

The game under consideration here in case (ii) can be considered as the one without initial probabilities of winning the prize, mentioned in the preceding paragraph. Thus, in case (ii), \( X^D \) must be equal to \( X^N \). This, together with the second condition in part (b) that \( X^N \leq M \), yields that \( X^D \leq M \). Clearly, this contradicts that \( X^D > M \), which is obtained above.
References


