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Abstract

This paper uses an otherwise standard, competitive growth model without externality and distortions to establish multiple balanced growth paths. Our model is based on the standard one-sector, endogenous growth model of Romer (1986), with a twist that households' preference depends partly upon how his/her consumption compares to a habit stock formed by his/her own past consumption. This model establishes multiple equilibria because habit persistence in preference induces an intertemporal complementarity effect among consumption flows, with current consumption reinforcing future consumption. As a result, there exist two balanced-growth paths, with one path exhibiting low consumption and habits and high economic growth, and the other exhibiting high consumption and habits and low growth, and thus a development trap. Both steady states are saddle points, but an initial condition cannot pin down the steady state to which an economy converges. Both steady states cannot be pareto-ranked because of no market failure.

Keywords: habit persistence; multiple equilibria.

JEL Classification: O41.

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1. Introduction

Recently there has been interest in economic models with multiple steady states. This strand is motivated by several empirical studies that document the existence of multiple development clubs (e.g., Baumol and Wolff, 1988; Quah, 1996; Durlauf and Quah, 1999). In accordance with the facts are models in economic development that display poverty traps, where economies with low initial capital stocks or incomes converge to a steady state with low per capita income, while economies with high initial capital stocks converge to a different steady state that corresponds to high per capita income.

The existing literature in models of economic development with multiple steady states may be broadly classified into three lines. One line concerns models of industrialization with externality in a modern technology, in which the productivity of a modern technology is enhanced by the number of firms using the technology, and an economy with low initial income chooses to use the traditional technology and is thus trapped in a steady state with low per capital income (e.g., Matsuyama, 1991; Krugman, 1991; Chen and Shimomura, 1998; Chen, Mo and Wang, 2002). A second line examines models of fertility choices and a tradeoff between the time allocated to bearing and rearing children and the time devoted to educating children, with the result that households find a large number of children in their best interests, leading to an equilibrium with low per capital income (e.g., Becker, Murphy and Tamura, 1990; Galor and Weil,1996; Palivos, 2001). Finally, a third line investigates costly intermediation in which a participation externality, where the cost of financial intermediation depends negatively on the mass of consumers, generates multiple steady states (e.g., Cooper and Ejarque, 1995; Becsi, Wang and Wynne, 1999). All these studies incorporate some forms of externality, which generates multiple equilibria through an *external* complementarity effect among households/investors.[1]

This paper uses a model to establish multiple steady states without relying on externality. Our model is based on an otherwise standard competitive, one-sector, endogenous growth model (e.g., Romer, 1986),

[1] In a two-country trade model by Shimomura (2004), multiple steady states are obtained not based on externality, but upon negative income effects so that the durable good is a Giffen good for a range of its shadow price.

with a departure that households' preference depends partly upon how his/her consumption compares to a habit stock formed by his/her own past consumption. This model creates multiple equilibria for the following reasons. With habit affecting preferences, high/low future consumption is expected to be associated with high/low current consumption, in order to attain a given level of utility for a household. This effect induces an *internal*, intertemporal complementarity effect among consumption flows, with current consumption reinforcing future consumption. More specifically, for a given initial state an agent may choose optimally to consume more or little now. As high/low current consumption forms habits quickly/slowly, s/he has to consume more/little in the future to obtain a proper consumption level in comparison to a high/low habit stock, in order to obtain the desired utility level. As a result, there are multiple equilibrium paths for consumption, with high/low future consumption expectations leading to high/low current consumption choices, all consistent with expectations. Consequently, the habits are formed quickly/slowly and capital stock is accumulated slowly/quickly in the equilibrium path associated with high/low consumption. As an effect of capital accumulation, the economic growth rate is low/high.

The two steady states are saddle points. For given initial capital and habit stocks, however, the initial history cannot pin down which steady state to converge, and the economy could converge to any of the two steady states depending upon the choices of consumption level. In the neighborhood of a steady state, a small disturbance leads the economy to shift *locally* to a new steady state around the original steady state, if the shadow habit price responds quickly. However, if the consumption responds faster than the shadow habit price in the face of a small disturbance, the equilibrium will globally converge to a steady state with low or high growth no matter where the original steady state is. These features differentiate the results of our model from the aforementioned studies with regard to two steady states. In some of these works, the steady states are both saddle points and there exists a threshold level that an initial condition determines the steady state to which an economy converges. In the reaming works, one of the steady states is a sink and the other is a saddle point, and *globally*, for most given initial conditions an economy converges to the steady state that is a sink. Moreover, the steady states in existing works can be pareto-ranked because of market failures in their models. The steady states cannot be pareto-ranked in our model because of no market failures.

Finally, the idea of habit persistence in preferences is hardly new, and dates to Marshall (1898) and

Dusenbery (1949). Recent years have seen applications in macroeconomics. Literature on asset pricing under habit persistence in preferences has been developing; see Abel (1990), Constantinides (1990) and Campbell and Cochrane (1999). The habit-persistence-in-preference feature has also been adopted in business cycle models, in order to explain some stylized features in business cycles (Boldrin, Christiano and Fisher, 2001); to demonstrate how procyclical tax policy affects the economy countercyclically (Ljungqvist and Uhlig, 2000); and to improve the responses of both spending and inflation to monetary-policy actions (Fuhrer, 2000). Finally, the habit persistence feature has been employed in endogenous growth models to obtain the result that increases in growth can cause increased savings (Carroll, Overland and Weil, 2000).[2]

As developed below, the theoretical model will be presented in the next section. Section 3 studies balanced-growth paths and transitional dynamics, while Section 4 examines the effects of two disturbances in relation to habit persistence. Concluding remarks will be made in Section 5.

2. Basic Model

Our basic model draws on Romer (1986) and Carroll, Overland and Weil (2000). Consider an economy populated by households and firms. There exists a continuum of infinite-lived, identical households, with no population growth. There also exists a continuum of representative firms, and households own the shares. It follows that the economy is a world of a representative household-producer.

2-1. Environment

The representative household is assumed to possess the following discounted, lifetime utility

$$U = \int_0^\infty e^{-\rho t}\, u\left(\frac{c(t)}{S^{\gamma}(t)}\right) dt, \quad \rho > 0,\ 0 < \gamma < 1, \tag{1a}$$

where u is the felicity function in t, $c(t)$ is the instantaneous private consumption flow in t, and $S(t)$ is the

[2] A related class of the models with intertemporal dependence in tastes is models with endogenous time preference rates pioneered by Uzawa (1968) and Wan (1970). See Shi and Epstein (1993) for a model that includes both habit formation and a variable time-preference rate, and Palivos, Wang and Zhang (1997) for a model of endogenous growth with an endogenous rate of time preference.

habitual stock. Parameter γ indexes the importance of habits. If $\gamma = 0$ then only the absolute level of consumption is important; while if $\gamma = 1$, then consumption relative to the habitual stock is what matters. For values of γ between 0 and 1, both the absolute and relative levels are important. For this study, we assume $0 < \gamma < 1$ so that absolute consumption level is not a consumer's concern. To facilitate the analysis, we have adopted a parametric felicity by assuming the following CES functional form

$$u\left(\frac{c(t)}{S^{\gamma}(t)}\right) = \frac{1}{1-\sigma}\left[\left(\frac{c(t)}{S^{\gamma}(t)}\right)^{1-\sigma} - 1\right], \ \sigma > 1. \tag{1b}$$

Assumption $\sigma>1$ is crucial for the results of multiple interior steady states. The presumption rules out the logarithmic felicity form, requiring one with curvature steeper than a logarithmic form. Restriction $\sigma>1$ is consistent with most empirical findings that intertemporal elasticity of substitution is smaller than one.

The stock of habits is assumed to evolve according to

$$\dot{S} = Bc^{\mu}(t)S^{1-\mu}(t) - \delta_s S(t), \quad 1/2 < \mu \leq 1, \ S(0) > 0 \ \textit{given}, \tag{2}$$

in which $B>0$ represents a technology coefficient that forms past consumption flows into habits, with $\delta_s \geq 0$ describing how existing habitual stock depreciates. Thus, a current consumption flow generates a long-lasting effect in a manner summarized by the stock of habits.

Equation (2) is more general than the specification in Carroll, et al. (2000). Here we follow other authors, such as Campbell and Cochrane (1999) and Lettau and Uhlig (2000) to assume that consumption accumulates future habits in relation to existing habitual stocks. More specifically, we assume that current consumption flows contribute toward future habit formation, possibly in association with existing habits; therefore, restriction $\mu \leq 1$ is made,[3] and to be consistent with a perpetual growth framework the formation technology is assumed to be of constant returns with respect to existing habits and current consumption. Moreover, restriction $\mu > 1/2$ is assumed so that current consumption contributes to forming future habitual

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[3] While habitual stock is $S(t) = \{[S(0)e^{-\delta_s t}]^{\mu} + B\mu\int_0^t [e^{-\delta_s(t-\tau)}c(\tau)]^{\mu}d\tau\}^{1/\mu}$ when $\mu \leq 1$, it is $S(t) = S(0)e^{-\delta_s t} + B\int_0^t e^{-\delta_s(t-\tau)}c(\tau)\,d\tau$ when $\mu = 1$.

stock more than current habits do. This formulation implies that the marginal productivity of consumption flows in forming habitual stocks is $B\mu$, which is less than or equal to B. It should be remarked that when $B=\delta_s=0$, past consumption does not form the habitual stock. Then, the model is reduced to conventional one-sector, endogenous growth models (e.g., Romer, 1986).

The representative firm is assumed to own the following production technology

$$y(t) = Ak(t),\ k(0) > 0 \ given, \tag{3}$$

where $y(t)$ is output, $k(t)$ is capital stock, and $A>0$ is a parameter. The technology is abstracted from labor so that capital stock should be interpreted broadly to include physical, as well as human capital (e.g., Rebelo, 1991).

Finally, as households own firms' shares, the representative household's budget constraint is

$$\dot{k} = y(t) - c(t) - \delta_k k(t), \tag{4}$$

where $\delta_k \geq 0$ is capital's depreciation rate. This equation says that disposable income, not consumed currently, becomes savings, which augments capital.

2-2. Optimization

The representative household's problem is to choose consumption, in order to maximize its discounted, lifetime utility (1a) and (1b), subject to habitual stock formation (2), production technology (3) and budget constraints (4), taking existing capital stock $k(t)$ and habits $S(t)$ as predetermined. To solve the dynamic optimization problem, we define the following current-value Hamiltonian

$$H(c,\ k,\ S,\ \lambda_k\ \ \lambda_s) = \frac{1}{1-\sigma}[(\frac{c}{S^{\gamma}})^{1-\sigma} - 1] + \lambda_k[Ak - c - \delta_k k] - \lambda_s[Bc^{\mu}S^{1-\mu} - \delta_s S],$$

where λ_k and λ_s denote the costate variables associated with (2) and (4), respectively. We should note that the shadow price of habit stock, $-\lambda_s$, is negative. The negative shadow price on habit stocks reflects the fact that a greater initial habit stock yields a lower utility flow for (1). Without abuse of terminology, in what follows we call the absolute value of the shadow price of habit stocks, λ_s, as the shadow habit price. Although under $\sigma>1$ the felicity is strictly concave in c(t), for $0<\gamma\leq 1$ it is not concave in $S(t)$ as a higher existing habit stock lowers utility. Since felicity function u is not concave in c and S, the Mangasarian sufficient theorem cannot be used. Instead, we need to apply the Arrow sufficient theorem to guarantee the concavity of the Hamiltonain (see Arrow and Kurz, 1970). Denote

$$\hat{H}(k, S, \lambda_k, \lambda_s) = \underset{\{c\in R^+\}}{Max} H(c, k, S, \lambda_k, \lambda_S).$$

In Appendix 1 we have shown that when $\mu=1$, under a mild condition $\hat{H}(k, S, \lambda_k(t), \lambda_s(t))$ is concave in k and S for fixed values of λ_k, λ_k and t.[4] For Case $\mu<1$, as long as μ is large enough, by continuity a similar condition assures $\hat{H}(k, S, \lambda_k(t), \lambda_s(t))$ to be concave in k and S for fixed values of λ_k, λ_k and t. Therefore, $k^*(t)$, $S^*(t)$ and $c^*(t)$ solve problem (1)-(4). As $\hat{H}(k, S, \lambda_k(t), \lambda_s(t))$ is not strictly concave in k and S, therefore $k^*(t)$, $S^*(t)$ and $c^*(t)$ are not necessarily unique.

Denote ρ as the time-preference rate. Applying the Pontryagin maximum principle, we get the following first-order conditions

$$\left(\frac{c}{S^{\gamma}}\right)^{1-\sigma}\frac{1}{c} = \lambda_k - \lambda_s\frac{B\mu S^{1-\mu}}{c^{1-\mu}}, \tag{5a}$$

$$A - \delta_k = \rho - \frac{\dot{\lambda}_k}{\lambda_k}, \tag{5b}$$

$$-\left(\frac{c}{S^{\gamma}}\right)^{1-\sigma}\frac{\gamma}{\lambda_s S} + \delta_s - B(1-\mu)\frac{c^{\mu}}{S^{\mu}} = \frac{\dot{\lambda}_s}{\lambda_s} - \rho. \tag{5c}$$

together with (2) and (4), and the transversality condition[5]

$$\underset{t\to\infty}{Lim}\ e^{-\rho t}H(t) = 0. \tag{5d}$$

Equation (5a) equates the marginal utility of current consumption to the marginal costs of foregone savings, net of the effect via the habit formation. Conditions (5b) and (5c) are two Euler equations that

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[4] The condition is $\frac{1}{\lambda\delta_S} + \frac{B}{\delta_S} > (1-\frac{1}{\sigma})\frac{(1-\gamma)+\gamma/\sigma}{(1+\gamma)(\sigma-1)^2+\gamma} + 1.$

[5] On the transversality condition in infinite horizon, optimal problems, see Michel (1982).

equate the net marginal productivity of capital stock, as well as the marginal utility of habitual stock, respectively, net of depreciation, to the time-preference rate, net of their respective capital gains (or losses).

3. Balanced Growth Paths and Transitional Dynamics

We are now ready to analyze the competitive market equilibrium.

Definition. A perfect-foresight equilibrium is a tuple $\{\frac{\dot{S}}{S(t)}, \frac{y(t)}{S(t)}, \frac{\dot{k}}{k(t))}, \frac{c(t)}{S(t)}, \frac{k(t)}{S(t)}, \frac{\lambda_s(t)}{\lambda_k(t)}\}$ that satisfies

(i) habitual stock formation; i.e., (2);

(ii) production technology; i.e., (3).

(iii) households' budgets; i.e., (4);

(iv) representative household-firm optimization; i.e., (5a)-(5d).

To analyze the market equilibrium, we transform the economic system into a 3x3 system in three variables $\{x, \lambda, z\}$, where $x \equiv \frac{c}{S}$, $\lambda \equiv \frac{\lambda_s}{\lambda_k}$ and $z \equiv \frac{k}{S}$. As we will see, our 3x3 system does not involve the second-order time derivatives of consumption when we keep the relative shadow price in the system, and is more simplified than the method utilized in Carroll, et al. (2000). This solution method is in line with the solution method used in Obtsfeld (1990) and Benhabib and Perli (1994), among others.

In order to derive the three-variable system, first we divide (5a) by (5c) to obtain

$$\frac{\dot{\lambda}_s}{\lambda_s} = \delta_s + \rho - \gamma \frac{x}{\lambda} - B[1-\mu+\mu\gamma]x^{\mu}. \tag{6a}$$

Next, differentiating (5a) with respect to time yields

$$(\frac{\dot{\lambda}_k}{\lambda_k} + \gamma(1-\sigma)\frac{\dot{S}}{S} + \sigma\frac{\dot{c}}{c}) + B\mu\frac{\lambda}{x^{1-\mu}}\{\frac{\dot{\lambda}_s}{\lambda_s} + [\gamma(1-\sigma)+(1-\mu)]\frac{\dot{S}}{S} - (1-\sigma-\mu)\frac{\dot{c}}{c}\} = 0,$$

which, together with (2), (5b) and (6a), can be rewritten as

$$\frac{\dot{c}}{c}=\frac{(A-\delta_k-\rho)+[\gamma(1+\frac{\sigma-1}{\mu})\frac{x}{\lambda}-(\rho+\mu\delta_s+\gamma(\sigma-1)\delta_s)+\gamma\mu Bx^{\mu}]B\mu\frac{\lambda}{x^{1-\mu}}+\gamma(\sigma-1)(B^2\mu\frac{\lambda}{x^{1-2\mu}}-\delta_s)}{\sigma+(\sigma+\mu-1)B\mu\frac{\lambda}{x^{1-\mu}}}\quad, \tag{6b}$$

Notice that since (A-δ_k) is the marginal product of capital, expression $\frac{\dot{c}}{c}$ reduces to the formulation in existing one-sector, endogenous growth models when B=γ=0, with the long-run, intertemporal elasticity of substitution equal 1/σ. With B>0 and γ>0 for habit formation, (6b) suggests that in the long-run, intertemporal elasticity of substitution is $[\sigma+(\sigma+\mu-1)B\mu\frac{\lambda}{x^{1-\mu}}]^{-1}$, which for σ>1is smaller than 1/σ, even for μ=1. This result is in contrast to the finding of the long-run, intertemporal elasticity being larger than 1/σ in an economy with habit formation in Carroll, et al. (2000) which resorts to the second-order time derivatives of c in order to derive the transitional dynamics of their economic system. Intuitively, current consumption forms habit stock that complements future consumption, thus lowering intertemporal elasticity of substitution for consumption.

Therefore, we have:

Proposition 1. *An economy with habit formation has a lower, long-run, intertemporal elasticity of substitution for consumption than one without habit formation*.

Finally, the economic system cannot be analyzed without transforming the growing variables into great ratios. We take differences between (6b) and (2), (6a) and (5b), and finally, (4) and (2) to obtain the following 3x3 economic system

$$\frac{\dot{x}}{x}=\frac{(A-\delta_k-\rho)+[\gamma(1+\frac{\sigma-1}{\mu})\frac{x}{\lambda}-(\rho+\mu\delta_s+\gamma(\sigma-1)\delta_s)+\gamma\mu Bx^{\mu}]B\mu\frac{\lambda}{x^{1-\mu}}+\gamma(\sigma-1)(B^2\mu\frac{\lambda}{x^{1-2\mu}}-\delta_s)}{\sigma+(\sigma+\mu-1)B\mu\frac{\lambda}{x^{1-\mu}}}-Bx^{\mu}+\delta_s, \tag{7a}$$

$$\frac{\dot{\lambda}}{\lambda}\equiv\frac{\dot{\lambda}_s}{\lambda_s}-\frac{\dot{\lambda}_k}{\lambda_k}=\delta_s-\gamma\frac{x}{\lambda}-B[1-(1-\gamma)\mu]x^{\mu}+A-\delta_k, \tag{7b}$$

$$\frac{\dot{z}}{z}\equiv\frac{\dot{k}}{k}-\frac{\dot{S}}{S}=A-\frac{x}{z}-\delta_k-Bx^{\mu}+\delta_s. \tag{7c}$$

Typically, the three-variable system is difficult to analyze. This is not the case here due to the block-recursive nature of the system. Capital to habit ratio, z, enters the system only through equation (7c); the other two equations form a separate subsystem in x and λ. Thus, while consumption and thus x affects physical capital accumulation and thereby capital to habit ratio, the ratio of capital to habits affects lifetime utility and consumption only by determining the initial optimal choices of these variables. Once these initial choices are made, λ and x evolve automatically. More specifically, while (7a) and (7b) jointly determine $\{\lambda(t), x(t)\}$, (7c) determines $z(t)$. When the three-variable system is solved, all other endogenous variables can be subsequently determined: $\frac{\dot{S}}{S(t)}$ is determined by (2), $\frac{y(t)}{S(t)}$ is determined by (3), $\frac{\dot{k}}{k(t)}$ is determined by (4), and finally, the consumption growth rate, $\frac{\dot{c}}{c(t)}$, is determined by (6b). Therefore, the market equilibrium can be characterized by analyzing system (7a)-(7c).

3-1. Balanced Growth Paths

We now determine the equilibrium in steady state. A balanced-growth path (BGP) is a steady-state, competitive market equilibrium, in which all growing variables grow at a constant rate over time. Therefore, $\dot{x} = \dot{\lambda} = \dot{z} = 0$ in the steady state. We use the conventional method to determine the balanced-growth path by investigating the shape of Loci summarized by (7a)-(7c) in two planes.

First, we start with Locus $\dot{\lambda}=0$ in a (λ, x) plane as it is less complicated. Differentiating (7b) with respect to x and λ, evaluated at $\dot{\lambda}=0$, the slope of Locus $\dot{\lambda}=0$ is

$$\left.\frac{dx}{d\lambda}\right|_{\dot{\lambda}=0} = -\frac{a_{22}}{a_{21}} > 0, \tag{8a}$$

where $a_{21} = -\gamma - [1-(1-\gamma)\mu]B\mu\frac{\lambda^*}{x^{*1-\mu}} < 0, \; and \; = -\gamma - \gamma B\mu\lambda^* < 0, \; if \; \mu=1,$

$a_{22} = \gamma\frac{x^*}{\lambda^*} > 0.$

Thus, Locus $\dot{\lambda}=0$ is upward sloping. Moreover, we have shown that Locus $\dot{\lambda}=0$ starts from the origin and approaches horizontal line $x = e \equiv (\frac{A+\delta_s-\delta_k}{B[1-(1-\gamma)\mu]})^{\frac{1}{\mu}} > 0$, when λ approaches infinity. See Figure 1.

[Insert Figure 1 here]

Next, in order to analyze the shape of Locus $\dot{x}=0,$ we impose

Condition R: $(\sigma-1)(1-\gamma)\delta_s > \rho,$

which is easy to meet as the intertemporal elasticity of substitution has been documented smaller than one and the time preference rate is usually low. Note that for $\gamma<1$ Condition R implies $(\sigma-1)(1-\gamma\delta_s) > \rho$. Under Condition R there is a positive threshold separating the behavior for Locus $\dot{x}=0$ as follows.

If we differentiate Locus $\dot{x}=0$ with respect to λ and x, we obtain

$$\left.\frac{dx}{d\lambda}\right|_{\dot{x}=0} = -\frac{a_{12}}{a_{11}} \lessgtr 0, \; if \; x \gtrless \hat{x}, \tag{8b}$$

where $a_{11} = -B\mu\frac{x^{\mu}}{\Psi}\{[B\mu(1-\gamma+\frac{\sigma-1}{\mu})-\frac{\gamma(\sigma-1)}{B\mu x^{1-\mu}}](2\mu-1)\frac{\lambda}{x^{1-\mu}}+[(\sigma-1)(1-\gamma\delta_s)-\rho]\frac{\lambda}{x}+\sigma(1-\gamma)+\gamma(1-\mu)\}$

$$< 0, \;\; if \; \mu>1/2 \; and \; x \geq \underline{x} \equiv (\frac{\gamma(\sigma-1)}{B^2\mu[\mu(1-\gamma)+\sigma-1]})^{\frac{1}{1-\mu}},$$

$$= -B\frac{x}{\Psi}\{[B(\sigma-\gamma)-\frac{\gamma(\sigma-1)}{B}]\lambda+[(\sigma-1)(1-\gamma\delta_s)-\rho]\frac{\lambda}{x}+\sigma(1-\gamma)\}<0, \; if \; \mu=1, \; B^2(\sigma-\gamma)>\gamma(\sigma-1),$$

$$a_{12} = -B\mu\frac{x^{\mu}}{\Psi}[\rho-(\sigma-1)(1-\gamma)\delta_s+B(\sigma+\mu-1)(1-\gamma)x^{\mu})] \lessgtr 0, \; if \; x \gtrless \hat{x} \equiv (\frac{(\sigma-1)(1-\gamma)\delta_s-\rho}{B(\sigma+\mu-1)(1-\gamma)})^{\frac{1}{\mu}}>0,$$

$$= -B\frac{x}{\Psi}[\rho-(\sigma-1)(1-\gamma)\delta_s+B(\sigma)(1-\gamma)x)] \lessgtr 0, \; if \; \mu=1, \; x \gtrless \hat{x} \equiv \frac{(\sigma-1)(1-\gamma)\delta_s-\rho}{B\sigma(1-\gamma)}>0,$$

$$\Psi \equiv \sigma+(\sigma+\mu-1)B\mu\frac{\lambda}{x^{1-\mu}}>0, \; and \quad = \sigma+\sigma B\lambda>0, \; if \; \mu=1.$$

For a_{11}, the third term in the large braces is positive and the second term is also positive under Condition R. Under $\mu>1/2$ and $x \geq \underline{x}$, it suffices to assure that the first term in the large braces is positive, and as a consequence, $a_{11}<0$. We should note that even if $x < \underline{x}$, a_{11} is negative as long as the effects through the second and third terms dominate the first term. For a_{12}, under Condition R it could be negative or positive depending upon whether x is above or below threshold $\hat{x}$. Therefore, Locus $\dot{x}=0$ is negatively sloping if $x \geq \hat{x}$, and positively slopping if $x \leq \hat{x}$, and is thus not a monotonic locus as illustrated in Figure 1. For $x \geq \hat{x}$, we have shown that Locus $\dot{x}=0$ has an infinite slope and has the x axis as the asymptote when

x approaches infinity, and Line $x = \hat{x}$ as the asymptote when λ approaches infinity. Alternatively, for $x \leq \hat{x}$, Locus $\dot{x}=0$ intersects the x axis at the region [0, x_0], where $x_0 \equiv [\frac{A-(\delta_k+\rho)-[\sigma+\gamma(\sigma-1)]\delta_s}{B[\sigma(1-\gamma)+\gamma(1-\mu)]}]^{\frac{1}{\mu}} > 0$, with a positive slope at Point (0, x_0),[6] and approaches Line $x = \hat{x}$ when λ approaches infinity.

Given the shapes of Loci $\dot{\lambda}=0$ and $\dot{x}=0$ in Figure 1, as $e > \hat{x}$,[7] these two loci have two interior intersections.[8] Therefore, there exist two interior BGPs, as indicated by points H and L in the figure, with BGP H having higher consumption to habit ratio and higher price of habit to price of capital ratio, while BGP L having lower consumption to habit ratio and lower price of habit to price of capital ratio.

Finally, in order to compare the economic growth rate for the two BGPs, we analyze Locus $\dot{z}=0$. The slope of Locus $\dot{z}=0$ is

$$\left.\frac{dx}{dz}\right|_{\dot{z}=0} = -\frac{a_{33}}{a_{31}} > 0, \tag{8c}$$

where $a_{31} = -(1 + B\mu x^{*^{\mu-1}} z^*) < 0, \; and \; = -(1 + Bz) < 0, \; if \; \mu = 1,$

$a_{33} = \frac{x^*}{z^*} > 0,$

which is positively sloping and, moreover, is concave in a (x, z) plane. Therefore, BGP H is associated with a higher capital to habit stock ratio, while BGP L is associated with a lower capital habit stock ratio.

The economic growth rate can be rewritten using (4), along with (3), to obtain

$$\frac{\dot{k}}{k} = \frac{\dot{y}}{y} = A - \delta_k - \frac{x^*}{z^*}. \tag{9}$$

Therefore, $\frac{\dot{k}}{k}$, and thus $\frac{\dot{y}}{y}$, is decreasing in $\frac{x}{z}$. Notice that consumption to habits relative to capital to

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6 The slope is $\left.\frac{dx}{d\lambda}\right|_{\dot{x}=0,\, \lambda_0=0,\, x=x_0} = -\frac{\rho-(\sigma-1)(1-\gamma)\delta_s+B(\sigma+\mu-1)(1-\gamma)x_0}{\sigma(1-\gamma)+\gamma(1-\mu)} > 0$, as $x_0 < \hat{x}$.

7 The condition for $e > \hat{x}$ is $(1-\gamma)[(\sigma+\mu-1)(A-\delta_k)+\mu\delta_s]+\rho+(1+\gamma)\mu[(\sigma-1)(1-\gamma)\delta -\rho]>0$, which is always met under Condition R.

8 A third intersection is a degenerated steady state with zero consumption and shadow price of habits.

habits, $\frac{x}{z}$, is just the consumption to capital ratio, $\frac{c}{k}$. Since the long-run $\frac{x}{z}$ in BGP H is $\frac{x_H}{z_H}$, which is smaller than $\frac{x_L}{z_L}$ in BGP L, the long-run, economic growth rate in BGP H is thereby larger than in BGP L. Since relationship $\frac{x_H}{z_H} > \frac{x_L}{z_L}$ implies $\frac{c_H}{k_H} < \frac{c_L}{k_L}$, consumption to capital ratio is lower for BGP H. As lower consumption in BGP H leads to slower habit accumulation, the price of habit is thus higher in BGP H, resulting in $\lambda_H > \lambda_L$ in Figure 1.

In summary,

Proposition 2. *There exists two BGPs in an economy with habit formation, with one BGP having a higher economic growth rate and the other having a lower economic growth rate.*

3-2. Dynamics

We now proceed to analyze the dynamic properties of the three-variable system by examining its transitional dynamics. The transitional dynamics of the economic system can be analyzed if we linearize the dynamic system of (7a)-(7c), evaluated at a BGP {x^*, λ^*, z^*}, to yield

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x - x^* \\ \lambda - \lambda^* \\ z - z^* \end{pmatrix}, \tag{10}$$

where (a_{ij})s are as defined in Section 3.

As the dynamic system of (7a)-(7c) involves one state variable, z, and two control variables, x and λ, there exists a unique equilibrium saddle path toward a BGP if the number of negative eigenvalues near the BGP is one, and there exists a continuum of equilibrium paths toward a BGP if the number of negative eigenvalues near the BGP is larger than one.

Denote **J** as the Jacobian matrix in (10) and θ as its eigenvalues. When we subtract matrix **J** from matrix θ**I**, where **I** is an identity matrix of order 3, then the eigenvalues are determined by equating determinant $|\mathbf{J}-\theta\mathbf{I}|$ to zero. If we expand $|\mathbf{J}-\theta\mathbf{I}|=0$, we obtain the following characteristic function

$$[\theta^2 - (a_{11}+a_{22})\theta + (a_{11}a_{22}-a_{21}a_{22})](\theta - a_{33}) = 0.$$

Solving the above polynomial function yields the following three eigenvalues,

$$\theta_1 = \frac{1}{2}[(a_{11}+a_{22}) - \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}\,] < 0,$$
$$\theta_2 = \frac{1}{2}[(a_{11}+a_{22}) + \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}\,] > 0,$$
$$\theta_3 = \frac{x^*}{z^*} > 0.$$

Given $a_{11} < 0$, $a_{21} < 0$ and $a_{22} > 0$, and in the neighborhood of BGP H, $a_{12} < 0$, we obtain $a_{11}a_{22} - a_{12}a_{21} < 0$. Then, $(a_{11}+a_{22}) < \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})} > 0$, and thus, $\theta_1 < 0 < \theta_2$ for BGP H. On the other hand, $a_{12}>0$ in the neighborhood of BGP L. As the slopes of both Loci $\dot{x}=0$ and $\dot{\lambda}=0$ are positive and the latter, $-\frac{a_{22}}{a_{21}}>0$, is steeper than the former, $-\frac{a_{12}}{a_{11}}>0$, we obtain $a_{11}a_{22} - a_{12}a_{21}<0$, and thus $\theta_1 < 0 < \theta_2$ for BG P L. Therefore, there is only one negative root around each BGP, and both BGPs are saddle points. Consequently, the dynamic growth path toward each BGP is unique. In Figure 2, Path DD is the unique saddle path toward BGP H, while Path FF is the unique saddle path toward BGP L.

[Insert Figure 2 here]

In general, in models with multiple steady states and all saddle points, history or predetermined state variables will govern the steady state to which the equilibrium moves. See Krugman (1991) and Matsuyama (1991) for discussion; however, this is not the case here. As an illustration, suppose that the initial state is at $z(0)$ in Figure 2. Then, there are two possible choices of consumption to habit ratios, at point A′ and B′, respectively. While point A′ stands for low consumption, low habit accumulation, high consumption to habit ratio and high habit price relative to capital price, point B′ stands for high consumption, high habit accumulation, low consumption to habit ratio and low habit price relative to capital price. Eventually, the equilibrium associated with point A′ moves toward BGP H with a high economic growth rate, while the equilibrium associated with point B′ moves toward BGP L with a low economic growth rate. Although the BGPs are saddle points in our model, history $z(0)$ alone is not able to pin down which BGP to move to.

The reasons for two possible equilibrium paths for an initial state are as follows. With the habit persistence in preferences, a household's higher current consumption is expected to lead to higher future consumption, in order for a consumption to habit ratio to attain a given level of utility. This effect induces an interaction among consumption flows, with current consumption reinforcing future consumption. When an agent expects to gain a certain utility level in the future, s/he may choose optimally to consume more now. As high current consumption forms more habits, s/he has to consume more in the future to have a proper consumption to high habit ratio in order to obtain the desired utility level. Alternatively, s/he could optimally choose to consume little now and in the future, in order to obtain the same utility level. As a result, there are two possible equilibrium paths for consumption, with high/low future consumption expectations leading to high/low current consumption choices, and all the equilibrium paths are consistent with expectations in equilibrium. Consequently, the habits are formed faster in the equilibrium path associated with high consumption than in the equilibrium path associated with low consumption, while capital stock is accumulated slowly in the former equilibrium path, and faster in the latter equilibrium path. As an effect of capital accumulation, the economic growth rate is low in transitional and steady state in the former equilibrium path, and high in the latter equilibrium path.

Summarizing the above result, we obtain

Proposition 3. *For any given initial capital and habit stock, there are a high and a low equilibrium consumption levels moving toward a BGP with a low and a high economic growth rate, respectively.*

4. Global Dynamics with Strong Habit Effects and Faster Habit Formation

We now analyze the dynamic equilibrium properties of small disturbances. We use as an illustrative example two disturbances in relation to habits: a more important habit effect (a higher γ) and a faster habit formation (a higher B). We have shown that both shocks shift Loci $\dot{\lambda}=0$ downward. While Locus $\dot{x}=0$ shifts upward when habits are more important for preferences (Figures 3 and 5), Locus $\dot{x}=0$ shifts downward, when consumption forms new habits faster (Figures 4 and 6). In addition, Locus $\dot{z}=0$ shifts downward in response to a faster habit formation (Figures 4 and 6). As there are local and global dynamics, we begin with a more important habit effect in preference and local dynamics, followed by a faster habit formation and local dynamics , and finally, global dynamics still later.

[Insert Figures 3-4 here]

4-1. When Preference Depends More on Habits: Local Dynamics

Local dynamic effects depend upon where the initial steady state is located. (The algebraic results near each steady state are reported in Appendix 2.) Suppose that the economy is originally at a high-growth equilibrium (i.e., BGP H in Figure 3). Then, higher dependence of consumption upon habits raises habit price to capital price ratio instantaneously if shadow habit price adjusts faster than consumption (see H^1 in Figure 3). In transition, consumption to habit ratio and habit price to capital price ratio both may increase or decrease, although habit price to capital price ratio is higher than the original ratio at BGP H. The changes in the capital to habit ratio can be seen from examining (7c) to obtain

$$a_{31}dx + a_{33}dz = a_{3\gamma}d\gamma + a_{3B}dB, \tag{11}$$

where a_{31}<0 and a_{33}>0 are in (8c), $a_{3\gamma}$=0, and $a_{3B} = x^{\mu} > 0$. Then, the effect upon capital to habit ratio is

$$\frac{dz}{d\gamma} = \frac{-a_{31}}{a_{33}}\frac{dx}{d\gamma}, \tag{11a}$$

which, under the fact $-a_{31}/a_{33}>0$, has the same sign as $dx/d\gamma$, and is thus ambiguous. We illustrate the case where consumption to habit ratio increases (path H^1H^γ in Figure 3), and therefore capital to habit ratio increases (path $H_zH^\gamma_z$ in Figure 3). Finally, if we use (9), the change in economic growth is

$$d(\frac{\dot{y}}{y}) = \frac{1}{z^*}(-dx + \frac{x^*}{z^*}dz). \tag{12}$$

Then, the effect upon economic growth of higher dependence of consumption upon habits is

$$\frac{d(\dot{y}/y)}{d\gamma} = \frac{1}{z^*}(-\frac{dx}{d\gamma} + \frac{x^*}{z^*}\frac{dz}{d\gamma}) = \frac{B\mu x^{\mu}}{a_{33}}\frac{dx}{d\gamma}, \tag{12a}$$

which has the same sign as $dx/d\gamma$, and is therefore ambiguous.

Alternately, suppose that the equilibrium is originally at a low-growth BGP (i.e., BGP L), then habit price to capital price ratio increases instantaneously in response to a higher complementarity between consumption and habits (see L^1 in Figure 3). In transition, consumption to habit ratio and habit price to capital price ratio both increase along path L^1L^γ. As consumption to habit ratio increases, capital to habit ratio must increase along path $L_zL^\gamma_z$ (cf. 11a). Moreover, as the growth effect has the same sign as the effect upon consumption to habit ratio (cf. 12a), economic growth is higher both in transition and in steady state.

4-2. When More Habits Can Be Formed by Consumption: Local Dynamics

Suppose first that the economy is originally at a high-growth equilibrium (i.e., BGP H in Figure 4). Then, when more habits are formed by a given consumption level, habit price to capital price ratio may increase or decrease instantaneously if habit price adjusts faster than consumption. We illustrate the case with a higher price ratio (see H^1 in Figure 4). In transition, consumption to habit ratio and habit price to capital price ratio both decrease toward BGP H^B. For the effect upon habit to capital ratio, using (11) we obtain

$$\frac{dz}{dB} = \frac{a_{3B}}{a_{33}} - \frac{a_{31}}{a_{33}}\frac{dx}{dB}, \tag{11b}$$

which under $a_{3B}/a_{33}>0$, has a direct positive effect, different from that of more important habit effects. However, as $a_{31}/a_{33}<0$ and $dx/dB<0$, there is an indirect negative effect through a lower consumption to habit ratio. As a result, upon the shock when consumption to habit ratio changes little initially, the direct effect dominates and capital to habit ratio increases. Over time, when consumption to habit ratio responds fully and if it decreases, the net effect is ambiguous. Finally, using (12) the effects upon the economic growth rate are

$$\frac{d(\dot{y}/y)}{dB} = \frac{1}{z^*}\left(-\frac{dx}{dB} + \frac{x^*}{z^*}\frac{dz}{dB}\right) = \frac{1}{z^*}\left(\frac{a_{3B}}{a_{33}} + \frac{B\mu z^*}{x^{*1-\mu}}\frac{dx}{dB}\right) = \frac{x^{*\mu}}{z^*}\left(1+\mu z^*\frac{B}{x^*}\frac{dx}{dB}\right). \tag{12b}$$

Again, when consumption to habit ratio changes little initially upon the shock, economic growth increases through the direct positive effect upon raising capital to habit ratio. Over time, as consumption to habit ratio changes more, the net growth effect becomes ambiguous.

Suppose now that the equilibrium is originally at a low-growth BGP (i.e., BGP L in Figure 4). When consumption can accumulate more habits, habit price to capital price ratio increases instantaneously (see L^1 in Figure 4). Over time, however, habit price to capital price ratio and consumption to habit ratio may both decrease or increase. In Figure 4 we illustrate the case where both ratios increase along path L^1L^B. Although there is a direct positive effect, the effect upon habit to capital ratio is ambiguous, due to an ambiguous indirect effect through consumption to habit ratio (cf. 11b). Similarly, in spite of a positive growth effect via the direct positive effect upon habit to capital ratio, the net growth effect is ambiguous, due to the ambiguous effect through consumption to habit ratio (cf. 12b).

4-3. Global Dynamics

Finally, the dynamics analyzed above is local, but we cannot rule out global dynamics. When habit effects become more important near the original high-growth BGP with initial capital to habit ratio at $z_H(0)$, consumption to habit ratio may react earlier than habit price to capital price ratio. If this is the case, consumption to habit ratio drops instantaneously from point H to H^2 in response to a habit disturbance (see Figure 5). Such a drop is then followed by increases or decreases in both habit price to capital price ratio and consumption to habit ratio. We illustrate the case where both ratios increase along path H^2L^γ. However, capital to habit ratio must decrease along path $H^2_zL^\gamma_z$.

[Insert Figure 5 here]

Similarly, when consumption accumulates more habits near the original high-growth BGP with initial capital to habit ratio at $z_H(0)$, consumption to habit ratio may react first and drop instantaneously from point H to H^2 in response to such a disturbance (in Figure 6). Then, reductions in or increases in both habit price to capital price and consumption to habit ratios may come later, with the case of both ratios decreasing along path H^2L^B exemplified in Figure 6. Again, capital to habit ratio must decrease along path $H^2_zL^B_z$.

[Insert Figure 6 here]

The global dynamics not only changes a BGP from high growth (H) to low growth (L^γ, L^B), but also from low growth (L) to high growth (H^{γ}, H^{B}). As an illustration, with the two aforementioned disturbances in the neighborhood of BGP L, when consumption adjusts faster than shadow habit price, the equilibrium jumps from L to L^2 instantaneously, with a higher consumption to habit ratio and a higher habit price to capital price ratio (see Figures 5 and 6). This may be followed by increases or decreases in consumption to habit ratio and habit price to capital price ratio. As an illustration, both ratios increases along path $L^2 H^{\gamma}$ or $L^2 H^{B}$, and as a result, capital to habit ratio increases along path $L^2_z H^{\gamma}_z$ or $L^2_{z\ Hbz}$.

Proposition 4.

(i) *Restricted to local dynamics, while faster habit formation has a positive growth effect in the short run and an ambiguous growth effect in the long run, a stronger habit effect raises economic growth when the economy is originally at a low-growth equilibrium.*

(ii) *Globally, when consumption to habit ratio adjusts faster than habit price to capital price ratio, both a stronger habit effect and faster habit formation could shift the economy to any one of the low- and high-growth equilibria no matter where the original steady state is located.*

5. Concluding Remarks

In this paper, we have analyzed a simple competitive, one-sector, endogenous growth model, that has been extended to allow for a utility function exhibiting habit persistence in preferences . The key feature in our model is that consumption has a long-lasting effect by forming habits. We use a standard solution procedure for a three-variable dynamic system without involving the second-order time derivatives for consumption in the system. The habit persistence in preferences brings forth a non-linear economic system resulting in two interior BGPs, with one exhibiting low consumption and habit formation and high economic growth, and the other displaying high consumption and habit formation and low economic growth and thus, a development trap.

The two steady states are saddle points, but for given initial capital and habit stocks the economy could converge to any of the two steady states depending upon the choices of consumption level. In the neighborhood of a steady state, a small disturbance leads the economy to shift to a new steady state around the original steady state, if the shadow habit price responds quickly. However, if consumption responds faster than the shadow habit price in the face of a small disturbance, the equilibrium may converge to any one of the two new steady states, no matter whether the original steady state is a low-growth one or a high-growth one. Moreover, the steady states cannot be pareto-ranked in our model because of no market failures.

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Appendix (Not Intended for Publication)

1 Proof of The Hamiltonian Satisfying the Arrow Sufficient Theorem

Maximizing the Hamiltonian with respect to *c*, i.e.,

$$\underset{\{c\}}{Max}\; H(c,\; k,\; S,\; \lambda_k\; \lambda_S) = \frac{1}{1-\sigma}[(\frac{c}{S^{\gamma}})^{1-\sigma}-1]+\lambda_k[Ak-c-\delta_k k]-\lambda_s[Bc^{\mu}S^{1-\mu}-\delta_s S],$$

leads to the following necessary and sufficient condition

$$H_c = \frac{c^{-\sigma}}{S^{\gamma(1-\sigma)}} - \lambda_k - \lambda_s[B\mu\frac{S^{1-\mu}}{c^{1-\mu}} - \delta_s] = 0, \quad \text{(A1a)}$$

$$H_{cc} = -\sigma\frac{c^{-(\sigma+1)}}{S^{\gamma(1-\sigma)}} + \lambda_s B\mu(1-\mu)\frac{S^{1-\mu}}{c^{-\mu}} < 0. \quad \text{(A1b)}$$

It is obvious (A1a) implies

$$\lambda_k + \lambda_s B\mu\frac{S^{1-\mu}}{c^{1-\mu}} = \frac{c^{-\sigma}}{S^{\gamma(1-\sigma)}} + \lambda_s\delta_s, \quad \text{(A1c)}$$

and leads to Relationship c=$c(S)$, that satisfies

$$\begin{aligned} c'(S) &= \frac{\gamma(\sigma-1)}{\sigma}\frac{c}{S} > 0,\; for\;\; \mu = 1, \\ &= \frac{1}{\Lambda}\left[\gamma(\sigma-1)\frac{c}{S} - \Omega\right] > 0,\; if\; \exists\; \underline{\mu} > 0,\; \ni \mu < \underline{\mu},\; for\;\; \mu < 1, \end{aligned} \quad \text{(A2a)}$$

where $\Omega = \frac{\lambda_s B\mu(1-\mu)}{\sigma}\frac{c^{\sigma+\mu}}{S^{\gamma(\sigma-1)+\mu}} > 0$ and $0 < \Lambda = 1 - \frac{\lambda_s B\mu(1-\mu)}{\sigma}\frac{c^{\sigma-1+\mu}}{S^{\gamma(\sigma-1)-1+\mu}} < 1$,

$$c''(S) = \frac{\gamma(\sigma-1)}{\sigma}(\frac{\gamma(\sigma-1)}{\sigma}-1)\frac{c}{S^2} < 0,\; for\;\; \mu = 1, \quad \text{(A2b)}$$

and

$$c''(S) = \frac{1}{\Lambda}\left\{\left[\frac{\gamma(\sigma-1)}{\sigma}\left(\frac{c'(S)}{S}-\frac{c}{S^2}\right)\right] - \frac{\lambda_s B\mu(1-\mu)}{\sigma}\left[(\sigma+\mu)\frac{c^{\sigma+\mu-1}}{S^{\gamma(\sigma-1)+\mu}}c'(S) - [\gamma(\sigma-1)+\mu]\frac{c^{\sigma+\mu}}{S^{\gamma(\sigma-1)+\mu-1}}\right]\right.$$
$$\left. - \frac{\gamma(\sigma-1)}{\sigma}\frac{c'(S)}{S}\frac{1}{\Lambda}\left[(\sigma+\mu-1)\frac{c^{\sigma+\mu-2}}{S^{\gamma(\sigma-1)+\mu-1}}c'(S) - (\gamma(\sigma-1)+\mu-1)\frac{c^{\sigma+\mu-1}}{S^{\gamma(\sigma-1)+\mu-2}}\right]\right\},\ if\ 0<\mu<1, \tag{A2c}$$

$$= \frac{1}{\Lambda}\frac{\gamma(\sigma-1)}{\sigma}\frac{1}{S}\left[c'(S)-\frac{c}{S}\right] - \frac{\Omega}{\Lambda}\frac{1}{c}\left[(\sigma+\mu)c'(S) - [\gamma(\sigma-1)+\mu]\frac{c}{S}\right]$$
$$- \frac{(1-\Lambda)}{\Lambda}\frac{c'(S)}{c}\left[(\sigma-1+\mu)c'(S) - [\gamma(\sigma-1)-1+\mu]\frac{c}{S}\right] < 0, if\ \exists\ \mu_0 > 0,\ \ni \mu < \mu_0.$$

For the terms in the three large brackets in (A2c), we have

$$c'(S) - \frac{c}{S} = \frac{\gamma(\sigma-1)-\sigma}{\sigma}\frac{c}{S} < 0,$$
$$(\sigma+\mu)c'(S) - [\gamma(\sigma-1)+\mu]\frac{c}{S} = \frac{\mu[\gamma(\sigma-1)-\sigma]c}{\sigma\Lambda S}\left(1+\lambda_s B(1-\mu)\frac{c^{\sigma+\mu-1}}{S^{\gamma(\sigma-1)+\mu-1}}\right) < 0,$$
$$(\sigma-1+\mu)c'(S) - (\gamma(\sigma-1)-1+\mu)\frac{c}{S} = \frac{(1-\mu)[\gamma(\sigma-1)-\sigma]c}{\sigma\Lambda S}\left(-1+\lambda_s B\mu\frac{c^{\sigma+\mu-1}}{S^{\gamma(\sigma-1)+\mu-1}}\right) \gtrless 0.$$

As in (A2a), by continuity there exists a $\mu_0>0$ large enough that the first term in (A2c), which is negative, dominates the positive second term and the ambiguous third term, and therefore $c''(S)$ in (A2c) is negative.

Substituting Relationship $c= c(S)$ into the Hamiltonain, we obtain a new Hamiltonian as follows,

$$\underset{\{S,\,k\}}{Max}\ \hat{H}(k,\ S,\ \lambda_k\ \ \lambda_S) = \frac{1}{1-\sigma}\left[\left(\frac{c(S)}{S^{\gamma}}\right)^{1-\sigma}-1\right] + \lambda_k[Ak-c(S)-\delta_k k] - \lambda_s[Bc(S)^{\mu}S^{1-\mu}-\delta_s S].$$

Then, we can derive the following conditions

$$\hat{H}_k = \lambda_k(A-\delta_k), \tag{A3a}$$

$$\hat{H}_{kk} = 0, \tag{A3b}$$

$$\hat{H}_{kS} = 0, \tag{A3c}$$

$$\hat{H}_S = \frac{c^{-\sigma}(S)c'(S)}{S^{\gamma(1-\sigma)}} - \gamma\frac{c^{1-\sigma}(S)}{S^{\gamma(1-\sigma)+1}} - \lambda_k c'(S) - \lambda_s B[\mu c^{\mu-1}(S)c'(S)S^{1-\mu} + (1-\mu)c^{\mu}(S)S^{-\mu}] + \lambda_s\delta_S. \qquad \text{(A3d)}$$

The condition for $\hat{H}_{SS}$ is more complicated, and we analyze it for (1) Case $\mu=1$, and (2) Case $\mu<1$.

1.1 For Case $\mu=1$

$$\begin{aligned}\hat{H}_{SS} = &-\sigma\frac{c^{-(\sigma+1)}(S)}{S^{\gamma(1-\sigma)}}[c'(S)]^2 + \frac{c^{-\sigma}(S)}{S^{\gamma(1-\sigma)}}[c''(S) - \gamma(\sigma-1)\frac{c(S)}{S}] \\ &- \gamma\frac{c^{-\sigma}(S)}{S^{\gamma(1-\sigma)+1}}\{(1-\sigma)c'(S) - [\gamma(1-\sigma)+1]\frac{c}{S}\} - (\lambda_k + \lambda_s B)c''(S).\end{aligned} \qquad \text{(A4a)}$$

If we substitute into (A2a)-(A2b) under $\mu=1$, (A4a) becomes

$$\hat{H}_{SS} = -\frac{\gamma}{\sigma}\frac{c^{-\sigma+1}}{S^{\gamma(1-\sigma)+2}}[(1+\gamma)(\sigma-1)^2 + \gamma\sigma] + [\frac{c^{-\sigma}}{S^{\gamma(1-\sigma)}} - (\lambda_k + \lambda_s B)]c''(S), \qquad \text{(A4b)}$$

which using (A1c) can be rewritten as

$$\begin{aligned}\hat{H}_{SS} &= -\frac{\gamma}{\sigma}\frac{c^{-\sigma+1}}{S^{\gamma(1-\sigma)+2}}[(1+\gamma)(\sigma-1)^2 + \gamma\sigma] - \lambda_s\delta_s\frac{\gamma(\sigma-1)}{\sigma}(\frac{\gamma(\sigma-1)}{\sigma} - 1)\ \frac{c}{S^2} \\ &= -\frac{\gamma}{\sigma}\frac{c}{S^2}\lambda_s\delta_s\left[\frac{c^{-\sigma}}{S^{\gamma(1-\sigma)}\lambda_s\delta_s}[(1+\gamma)(\sigma-1)^2 + \gamma\sigma] - (\sigma-1)[1 - \frac{\gamma(\sigma-1)}{\sigma}\right].\end{aligned} \qquad \text{(A4c)}$$

As $\frac{c^{-\sigma}}{S^{\gamma(1-\sigma)}\lambda_s\delta_s} = \frac{\lambda_k + \lambda_s B}{\lambda_s\delta_s} - 1 \equiv \frac{1 + \lambda B}{\lambda\delta_s} - 1$ according (A1c), where $\lambda \equiv \lambda_s/\lambda_k$ as defined in the text, then $\hat{H}_{SS} < 0$ in (A4c), if the following condition is satisfied

Condition S: $\frac{1}{\lambda\delta_S} + \frac{B}{\delta_S} > (1 - \frac{1}{\sigma})\frac{(1-\gamma) + \gamma/\sigma}{(1+\gamma)(\sigma-1)^2 + \gamma} + 1.$

Condition S is met if δ_S and λ are small, and B is large.

Then, the Hessian

$$\begin{vmatrix} \hat{H}_{SS} & \hat{H}_{Sk} \\ \hat{H}_{kS} & \hat{H}_{kk} \end{vmatrix} = 0. \qquad (A5)$$

Therefore, when μ=1, the Hamiltonian is concave and satisfies Arrow's sufficient theorem.

1.2 For Case $0 < \mu < 1$

$$\begin{aligned}\hat{H}_{SS} = & -\sigma\frac{c^{-(\sigma+1)}(S)}{S^{\gamma(1-\sigma)}}[c'(S)]^2 + \frac{c^{-\sigma}(S)}{S^{\gamma(1-\sigma)}}[c''(S) - \gamma(\sigma-1)\frac{c(S)}{S}] \\ & - \gamma\frac{c^{-\sigma}(S)}{S^{\gamma(1-\sigma)+1}}\{(1-\sigma)c'(S) - [\gamma(1-\sigma)+1]\frac{c}{S}\} - (\lambda_k + \lambda_S B\mu c^{\mu-1}S^{1-\mu})c''(S) \\ & - \lambda_S B\mu(1-\mu)c^{\mu}S^{-\mu}(\frac{1}{S} - \frac{1}{c})(\frac{S}{c}c'(S) - 1),\end{aligned} \qquad (A6)$$

whose first three terms are the same as in (A4a), the fourth term is also similar to (A4a) if μ is close to 1, and the last term is an extra term which is ambiguous and is small if μ is close to 1.

Given the similarity between form (A6) and form (A4a), by continuity there exists μ_1 large enough such that for all $\mu < \mu_1$, a condition similar to Condition S can guarantee (A6) to be negative, and thus the Hamiltonian to be concave in (S, k).

2 Derivation of the Comparative-Static Results When Only Local Dynamics Are Considered.

If we differentiate (7a)-(7c) around a steady state, we obtain

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}\begin{pmatrix} dx \\ d\lambda \\ dz \end{pmatrix} = \begin{pmatrix} a_{1\gamma} \\ a_{2\gamma} \\ 0 \end{pmatrix} d\gamma + \begin{pmatrix} a_{1B} \\ a_{2B} \\ a_{3B} \end{pmatrix} dB, \qquad (A7)$$

where a_{11}, a_{12}, a_{21}, a_{12}, a_{31} and a_{33} are as defined in the text, and

$$a_{1\gamma} = -\frac{x}{\Psi}\{[(\sigma-1)(Bx^{\mu}-\delta_s)+\mu Bx^{\mu}]+(\sigma-1)(Bx^{\mu}-\delta_s)B\mu\frac{\lambda}{x^{\mu}}+(B\mu)^2\frac{\lambda}{x^{1-2\mu}}\} < 0,$$

$$a_{1B} = \frac{x}{\Psi}\{(1-\gamma)(\sigma+\mu-1)(2Bx^{\mu}-\delta_s)\mu\frac{\lambda}{x^{1-\mu}}+[\rho+(1+\gamma)\mu\delta_s)]\mu\frac{\lambda}{x^{1-\mu}}+[\gamma(1-\mu)+\sigma(1-\gamma)]x^{\mu}\} > 0,$$

$$a_{2\gamma} = x + B\mu\lambda x^{\mu} > 0,$$

$$a_{2B} = [1-(1-\gamma)\mu]\lambda x^{\mu} > 0,$$

$$a_{3B} = x^{\mu} > 0.$$

2-1 In the neighborhood of BGP H, a_{12} <0, and we obtain

$$\frac{dx}{d\gamma} = \frac{a_{33}}{\Delta}(a_{1\gamma}a_{22} - a_{2\gamma}a_{12}) \gtrless 0,\ if\ \left.\frac{d\lambda}{d\gamma}\right|_{\dot{\lambda}=0} \lessgtr \left.\frac{d\lambda}{d\gamma}\right|_{\dot{x}=0},\ as\ a_{33} > 0\ and\ a_{1\gamma}a_{22} - a_{2\gamma}a_{12} \lessgtr 0,$$

$$\frac{dz}{d\gamma} = \frac{-a_{31}}{\Delta}(a_{1\gamma}a_{22} - a_{2\gamma}a_{12}) \gtrless 0,\ if\ \left.\frac{d\lambda}{d\gamma}\right|_{\dot{\lambda}=0} \lessgtr \left.\frac{d\lambda}{d\gamma}\right|_{\dot{x}=0},\ as\ a_{31} < 0,\ and\ a_{1\gamma}a_{22} - a_{2\gamma}a_{12} \lessgtr 0,$$

$$\frac{d(\dot{y}/y)}{d\gamma} = -\frac{d(x/z)}{d\gamma} = \frac{1}{z}\left(-\frac{dx}{d\gamma} + \frac{x}{z}\frac{dz}{d\gamma}\right) = \frac{-(a_{1\gamma}a_{22} - a_{2\gamma}a_{12})}{\Delta z}\left(a_{33} + a_{31}\frac{x}{z}\right)$$
$$= \frac{(a_{1\gamma}a_{22} - a_{2\gamma}a_{12})}{\Delta}B\mu\frac{x^{\mu}}{z} \gtrless 0,\ if\ \left.\frac{d\lambda}{d\gamma}\right|_{\dot{\lambda}=0} \lessgtr \left.\frac{d\lambda}{d\gamma}\right|_{\dot{x}=0},$$

$$\frac{dx}{dB} = \frac{a_{33}}{\Delta}(a_{1B}a_{22} - a_{2B}a_{12}) < 0,\ as\ a_{33} > 0,\ and\ a_{1B}a_{22} - a_{2B}a_{12} > 0,$$

$$\frac{dz}{dB} = \frac{a_{3B}}{a_{33}} + \frac{-a_{31}}{\Delta}(a_{1B}a_{22} - a_{2B}a_{12}) < \frac{a_{3B}}{a_{33}},\ as\ a_{31} < 0,\ and\ a_{1B}a_{22} - a_{2B}a_{12} > 0,$$

$$\frac{d(\dot{y}/y)}{dB} = -\frac{d(x/z)}{dB} = \frac{1}{z}\left(-\frac{dx}{dB} + \frac{x}{z}\frac{dz}{dB}\right) = \frac{x}{z^2}\frac{a_{3B}}{a_{33}} - \frac{(a_{1B}a_{22} - a_{2B}a_{12})}{\Delta z}\left(a_{33} + a_{31}\frac{x}{z}\right)$$
$$= \frac{x}{z^2}\frac{a_{3B}}{a_{33}} + \frac{(a_{1B}a_{22} - a_{2B}a_{12})}{\Delta}B\mu\frac{x^{\mu}}{z} < \frac{x}{z^2}\frac{a_{3B}}{a_{33}},$$

where $\Delta = a_{33}(a_{11}a_{22} - a_{21}a_{22}) < 0.$

2-2 In the neighborhood of BGP L, a_{12} >0, and we obtain

$$\frac{dx}{d\gamma} = \frac{a_{33}}{\Delta}(a_{1\gamma}a_{22} - a_{2\gamma}a_{12}) > 0,\ as\ a_{33} > 0\ and\ a_{1\gamma}a_{22} - a_{2\gamma}a_{12} < 0,$$

$$\frac{dz}{d\gamma} = \frac{-a_{31}}{\Delta}(a_{1\gamma}a_{22} - a_{2\gamma}a_{12}) > 0,\ as\ a_{31} < 0,\ and\ a_{1\gamma}a_{22} - a_{2\gamma}a_{12} < 0,$$

$$\frac{d(\dot{y}/y)}{d\gamma} = -\frac{d(x/z)}{d\gamma} = \frac{1}{z}(-\frac{dx}{d\gamma}+\frac{x}{z}\frac{dz}{d\gamma}) = \frac{-(a_{1\gamma}a_{22} - a_{2\gamma}a_{12})}{\Delta}(a_{33}+a_{31}\frac{x}{z}) = \frac{(a_{1\gamma}a_{22} - a_{2\gamma}a_{12})B\mu x^{*\mu}}{\Delta} > 0,$$

$$\frac{dx}{dB} = \frac{a_{33}}{\Delta}(a_{1B}a_{22} - a_{2B}a_{12}) \begin{matrix} > \\ < \end{matrix} 0,\ \left.\frac{d\lambda}{dB}\right|_{\dot{\lambda}=0} \begin{matrix} < \\ > \end{matrix} \left.\frac{d\lambda}{dB}\right|_{\dot{x}=0},\ as\ a_{33} > 0,\ and\ a_{1B}a_{22} - a_{2B}a_{12} \begin{matrix} < \\ > \end{matrix} 0,$$

$$\frac{dz}{dB} = \frac{a_{3B}}{a_{33}} + \frac{-a_{31}}{\Delta}(a_{1B}a_{22} - a_{2B}a_{12}) \begin{matrix} > \\ < \end{matrix} \frac{a_{3B}}{a_{33}},\ if\ \left.\frac{d\lambda}{dB}\right|_{\dot{\lambda}=0} \begin{matrix} < \\ > \end{matrix} \left.\frac{d\lambda}{dB}\right|_{\dot{x}=0},\ as\ a_{31} < 0,\ and\ a_{1B}a_{22} - a_{2B}a_{12} \begin{matrix} < \\ > \end{matrix} 0,$$

$$\frac{d(\dot{y}/y)}{dB} = -\frac{d(x/z)}{dB} = \frac{1}{z}(-\frac{dx}{dB}+\frac{x}{z}\frac{dz}{dB}) = \frac{x}{z^2}\frac{a_{3B}}{a_{33}} - \frac{(a_{1B}a_{22} - a_{2B}a_{12})}{\Delta z}(a_{33}+a_{31}\frac{x}{z})$$
$$= \frac{x}{z^2}\frac{a_{3B}}{a_{33}} + \frac{(a_{1B}a_{22} - a_{2B}a_{12})}{\Delta}B\mu\frac{x^{\mu}}{z} \begin{matrix} > \\ < \end{matrix} \frac{x}{z^2}\frac{a_{3B}}{a_{33}},\ if\ \left.\frac{d\lambda}{dB}\right|_{\dot{\lambda}=0} \begin{matrix} < \\ > \end{matrix} \left.\frac{d\lambda}{dB}\right|_{\dot{x}=0},$$

where $\Delta = a_{33}(a_{11}a_{22} - a_{21}a_{22}) < 0.$

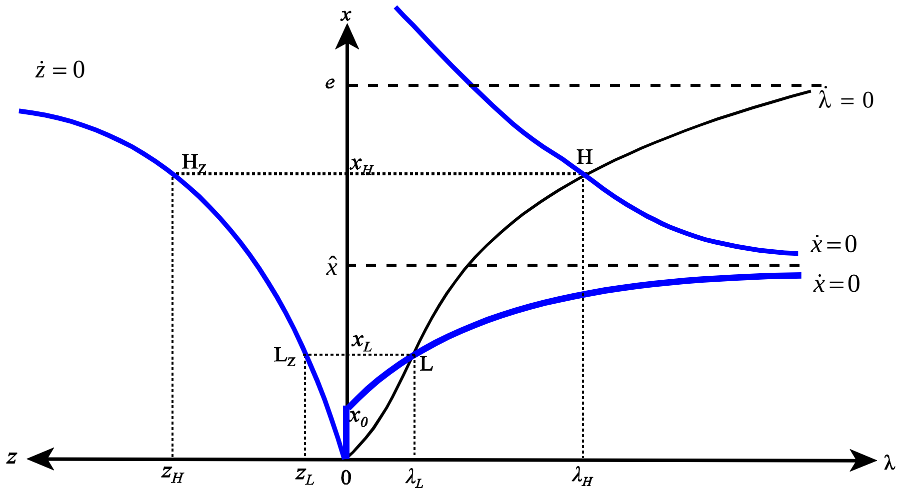

Figure 1. Balanced growth paths

Note: $x_0 \equiv [\frac{A-(\delta_k+\rho)-[\sigma+\gamma(\sigma-1)]\delta_s}{B[\sigma(1-\gamma)+\gamma(1-\mu)]}]^{\frac{1}{\mu}}$, $\hat{x} = [\frac{(\sigma-1)(1-\gamma)\delta_s-\rho}{B(\sigma+\mu-1)(1-\gamma)}]^{\frac{1}{\mu}}$, $e = [\frac{A+\delta_s-\delta_k}{B[1-(1+\gamma)\mu]}]^{\frac{1}{\mu}}$

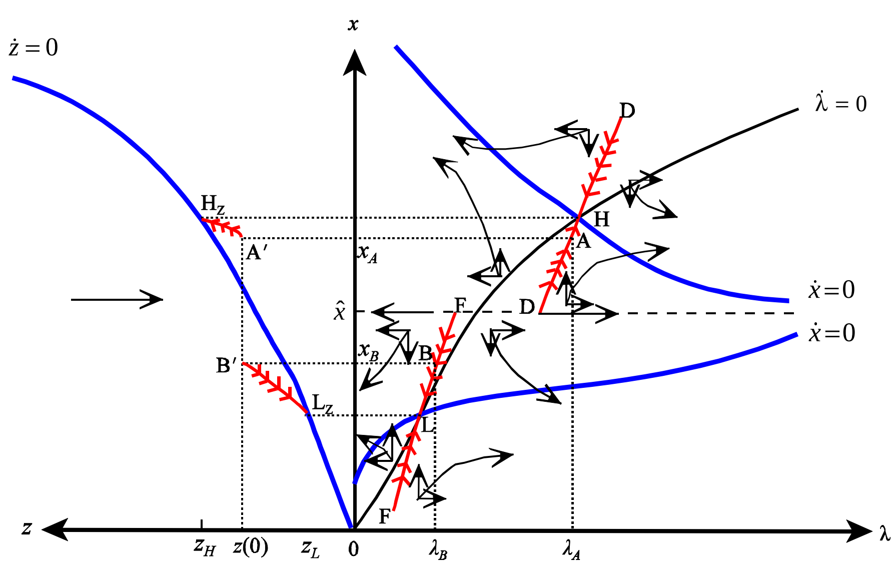

Figure 2. Transitional Dynamics

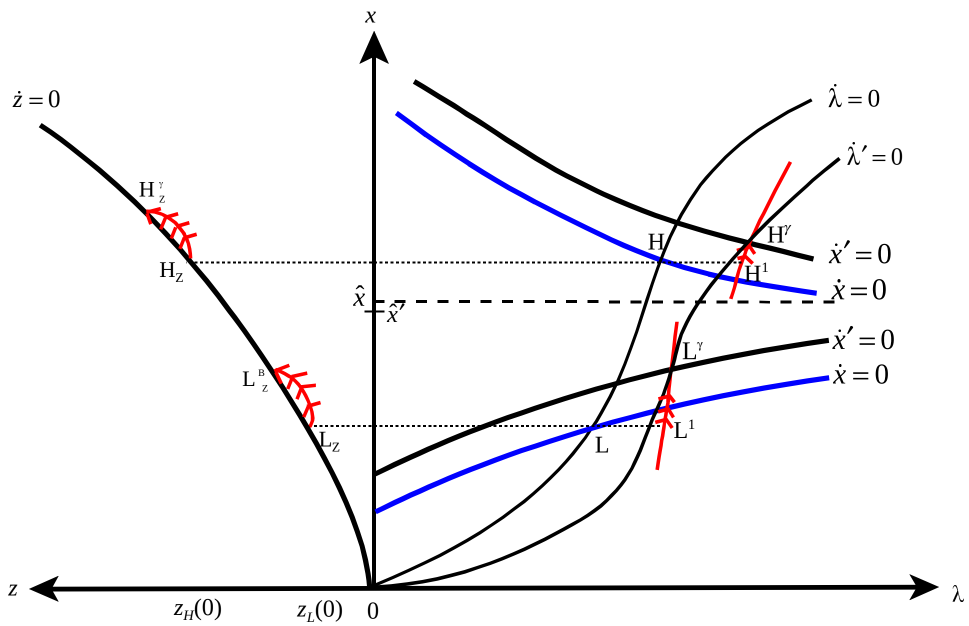

Figure 3. A More Important Habit Effect (Higher γ): Local Dynamics

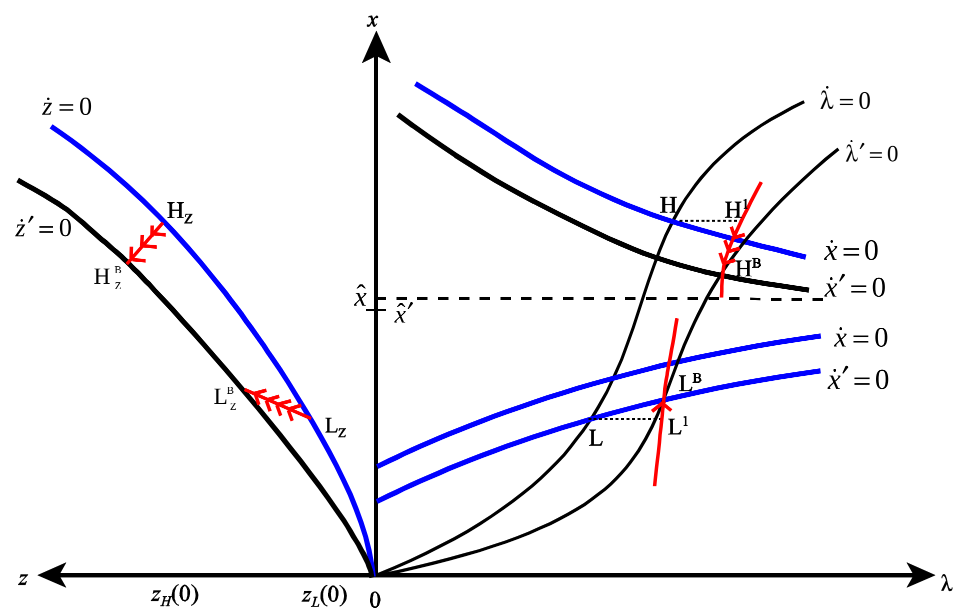

Figure 4. A Faster Habit Formation (Higher B): Local Dynamics

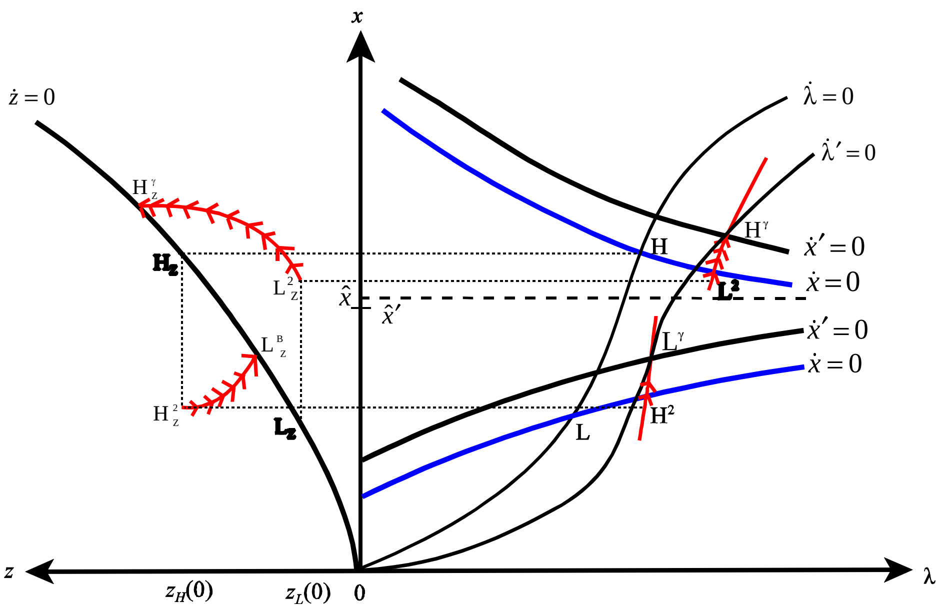

Figure 5. A More Important Habit Effect (Higher γ): Global Dynamics

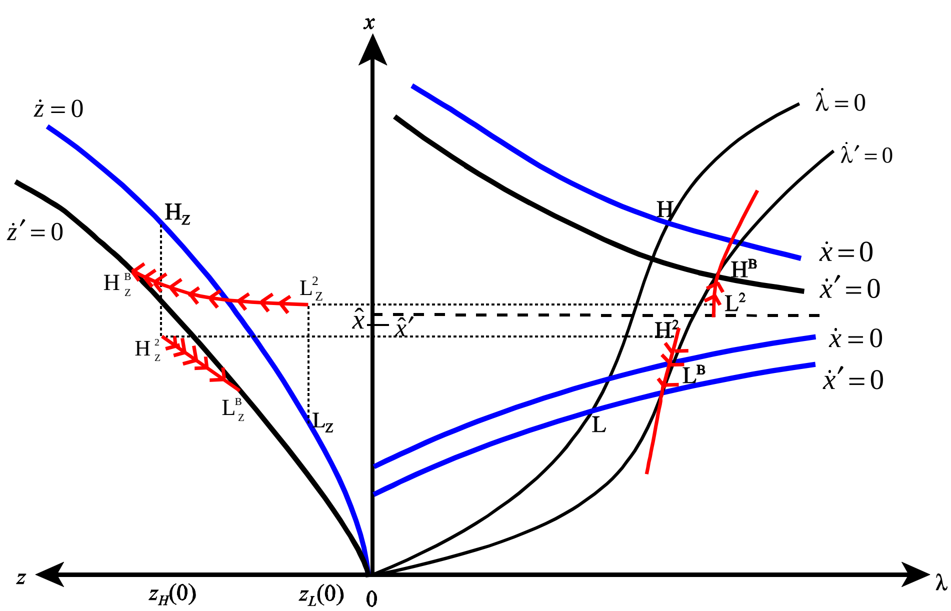

Figure 6. A Faster Habit Formation (Higher *B*): Global Dynamics

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