Optimal Disability Insurance and Unemployment Insurance With Cyclical Fluctuations

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Abstract

This paper studies the optimal joint design of disability insurance and unemployment insurance in an environment with moral hazard, when an individual’s health status is private information, taking into account cyclical fluctuations. I first show how disability benefits and unemployment benefits vary with aggregate economic conditions in an optimal contract that resolves this information problem. I then consider a calibrated version of the model and study the quantitative implications of changing from the current system to the optimal one. Last, in a special case, I demonstrate that the optimal joint insurance system can be implemented using a relatively simple model. In the optimal system, disability benefits are designed such that the system punishes workers who stay unemployed for a long time, reducing the unemployment rate by roughly 40 percent and incurring substantial cost savings from resolving incentive problems. Using the model to implement the optimal system, I am able to analyze in details the driving forces behind the differences between the current system and the optimal system. Under the optimal joint design of these insurance programs, disability insurance serves as an additional tool for the government to provide incentives for the job search.

Keywords: Disability insurance, Unemployment insurance, Business cycles, Optimal policy.

JEL Classification: D8, H5, J6.

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1 Introduction

Disability insurance (DI) and unemployment insurance (UI) are two important social insurance programs in the United States that provide relief when people suffer from income fluctuations. As of December 2013, 8.9 million disabled individuals received DI benefits, which corresponds to 5.7% of the population of workers in the labor force. The unemployment rate at the time was at 6.2%, which is higher than the pre-recession period, leaving 827 thousand workers who had exhausted their UI by the first quarter of 2014. While DI and UI both play significant roles in helping individuals to smooth their consumption, both programs are subject to incentive problems, which result in information problems. For example, job search efforts for unemployed workers are hardly monitored, making it difficult to motivate or observe their job search process, and approximately half of awards of DI go to applicants with difficult-to-verify disabilities, such as mental disorders and chronic pain, especially from back injuries (Autor and Duggan (2003)), making it tempting for the workers to overstate their health limitations. Further, these incentive problems respond differently to business cycles making them even harder to quantify but suggesting that the interrelation of the two requires further study. In fact, both DI and UI applications and awards are reported to be countercyclical. The cyclicality of SSDI applications and the unemployment rates are shown in Figure 1, and we can observe that SSDI applications surge a year or two after unemployment peaks after the reform of SSDI in 1984. Furthermore, it has been reported that SSDI awards have a similar pattern. If we think that health shocks are a-cyclical, why are the observed patterns of DI applications and awards countercyclical?

One of the possible explanations is that workers have more incentive to claim disability after long unemployment spells. To make this situation even worse, the current design of DI and UI does not consider the interrelation between DI and UI. In the United States, a typical unemployment insurance program provides 26 weeks of benefits, but extended unemployment insurance programs are adopted subsequently in recessions, providing some groups of people with different extended periods when their insurance benefits are exhausted. In the most recent recession, benefits were provided for a maximum of 99 weeks - however, these extensions were undertaken without considering disability insurance. In terms of disability insurance, its benefits are calculated with the Primary Insurance Amount (PIA) formula, which uses the inflation-adjusted averaged monthly income. The benefits are progressive in the sense that low-income people receive higher benefits, but the amounts of benefits do not vary with aggregate economic states.

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2See Mueller et al. (2016), Black et al. (2002), Autor and Duggan (2003), Duggan and Imberman (2009), and Coe et al. (2011) for details.
as unemployment insurance does. From the discussions above, we can observe that the patterns of disability insurance and unemployment insurance seem independent from each other. Further, the fact that the amounts of DI benefits do not vary in bad times gives workers incentives to claim disability if they are able to overstate their health limitations. Thus, studying their connection, as this paper proposes to do by analyzing the joint design problem while taking into account cyclical fluctuations, potentially provides greater insight into how to tackle these incentive problems.

To address the incentive problems associated with UI, DI, and business cycles, this paper studies the optimal policy design of disability insurance and unemployment insurance in an environment where job finding and separation rates fluctuate with aggregate economic states and workers can overstate their health limitations. The objective of this paper is three-fold. First, I build a model of UI, DI, and business cycles to characterize optimal contract design. Second, I calibrate the model to the U.S. data and calculate the inefficiency in the current social insurance system. I consider the potential cost savings from switching from the current system to the optimal system and conduct counterfactual analyses by presenting the impact of different policy reforms. This paper suggests that the potential benefits for policy makers are substantial if they adjust social insurance policies according to aggregate economic state changes and the joint design of DI and UI. Last, I consider a special case of the model that can be solved explicitly. This allows me to demonstrate the implementation of the optimal contract via simple instruments: equating worker’s self-insured behavior and optimal insurance. This implementation helps to obtain deeper insights from the optimal contracts and to understand the driving forces behind the substantial
savings obtained in the quantitative section.

To be specific, I build a tractable model that incorporates disability insurance, similar to Golosov and Tsyvinski (2006), into previous work by Williams and Li (2014) on unemployment insurance, which itself is based on Hopenhayn et al. (1997). In my model, similar to Williams and Li (2014), a risk-averse worker exerts costly job search efforts that increase job arrival rates, and job finding and separation rates depend on the aggregate economic climate. I model the business cycles by having job finding rates and job separation rates switch according an exogenous Markov process. This paper diverges from the preceding literature on unemployment insurance by adding a health shock that makes a worker incapable of working, similar to Golosov and Tsyvinski (2006). The government agency provides two kinds of insurance, unemployment insurance and disability insurance, and aims to minimize the cost of providing joint insurance while still providing insurance that satisfies the workers’ incentive constraints. The work incentive problem arises due to the fact that workers’ health conditions are unobservable to the agency in addition to their unobservable job search efforts and the fact that the job finding and separation rates depend on aggregate economic conditions.

By modeling DI and UI together with business cycles, our model enables us to develop the optimal design to account for when workers overstate their health status and how hard they search for jobs, and to compare the optimal social insurance schemes with the current programs through different aggregate economic states. While there are many studies of UI and, to a lesser extent, DI with respect to their independent effects on labor supply, few papers have studied how DI and UI jointly affect labor supply decisions with business cycles. To the best of my knowledge, no previous papers have developed an optimal policy design that considers the interaction between DI and UI with business cycles.

After laying out the model, I consider the quantitative implications of the model and study the impacts of DI on the unemployment insurance system. I first calibrate the model to obtain some key parameters. Then I study the following question, asking: what amount of cost reductions result when the worker switches from the current system to the optimal one? I found that the cost reductions could be substantial and that switching to the optimal system reduces unemployment rates by around 40%.

I then consider the impact of different policy reforms. In particular, I consider the impact of two policy reforms: (1) extending the maximum UI duration to 99 weeks in recessions, and (2) reconsidering the design of the optimal DI system but taking the current UI system as given. The first policy reform is motivated by the actions taken in the great recession by the US government where unemployed workers may have received UI for up to 99 weeks. I found that there is little difference in cost savings and the unemployment rate between the UI system with standard
UI duration (26 weeks) and the further-extended one (99 weeks). The second experiment is intended to show the interdependence between the DI and UI systems and the potential benefit to considering that interdependence when designing policy reforms. I found that even though the government can only change the DI system, and not the UI system, there will still be big differences in the cost savings and unemployment rate based on these changes.

Last, in a special case of the model, I consider the implementation of the optimal contracts using a worker’s consumption-saving-effort model, as in Williams and Li (2014). I find that the optimal contracts can be implemented by constant payments in each state through taxes while employed, subsidies while unemployed, and lump-sum transfers when a worker’s employment state (employed/unemployed/disabled, boom/recession) switches. The implementation of the model allows me both to understand the driving forces behind the optimal contracts and analyze the comparative statics through an individual agent’s borrowing and saving activities. The main finding when the joint insurance is considered is that disability insurance can serve as an additional tool for the government to provide incentives for individuals searching for a job, which explains why substantial cost savings can be obtained in the quantitative section.

The papers that are most closely related to this paper are Hopenhayn et al. (1997), Williams and Li (2014), and Golosov and Tsyvinski (2006). Hopenhayn et al. (1997) and Williams and Li (2014) focus on the optimal design of unemployment insurance, while Williams and Li (2014) add business cycles on top of Hopenhayn et al. (1997)’s model. In Golosov and Tsyvinski (2006), the authors study the optimal design of disability insurance and focus on the asset-testing implementation mechanism. The solvable case and implementation of the optimal contract in this study follows the ones in Williams and Li (2014)closely, but adding the disability state allows me to show the impact of UI on DI. The implications of this paper’s model are in line with the empirical literature. Among the papers that study the cyclicality of DI are Autor and Duggan (2003), Rutledge (2012), and Mueller et al. (2016). Autor and Duggan (2003) attribute the labor force exit propensity of displaced high school dropouts after 1984 to three major factors: a reduced screening stringency for DI, a decline in demand for less-skilled workers, and an unforeseen increase in the earnings replacement rate. They find that the sum of these forces can account for a decrease by one-half a percentage point in measured U.S. unemployment. Mueller et al. (2016) incorporate unemployment insurance to Autor and Duggan (2003) model by drawing on Rutledge (2012)’s model of UI and job search. Given that the cyclicality of DI applications and awards is consistently found to be a stylized fact, the findings of Mueller et al. (2016) imply that there should be other channels through which to explain how a surge of DI applications and awards closely follows business cycles. In this paper, I am able to structurally analyze the cyclicality patterns of DI and UI in an optimal design framework as well as explore the effects of earnings
replacement rates on workers from different income groups.

The rest of the paper is organized as follows. In section 2, I lay out the model. Then I study the optimal contract design in section 3 and demonstrate the quantitative implications of the optimal design in section 4. Section 5 studies the implementation of the contract. Finally, Section 6 concludes.

2 The Model

In this section, I lay out the model for this study. The model for this study is based on the UI and business cycle model in Williams and Li (2014) and adds disability shocks similar to Golosov and Tsyvinski (2006).

2.1 The Setup

I consider an infinitely lived agent (worker) who transitions between being employed and unemployed when healthy and could also become disabled. If employed, the agent receives a constant wage $\omega$. If unemployed, the agent earns no income, but he may exert effort to find a job, with that effort being costly to him but increasing the arrival rate of a job. When the agent becomes disabled, I assume that the job arrival rate and wage is zero and the disability is an absorbing state: i.e., the disabled worker will not become healthy again. In addition, I assume that the economy switches between booms and recessions. In a boom, the job finding rate is higher, while the separation rate is lower. I will use $s_t \in \{B, R\}$ to denote booms and recessions.

When unemployed and in a state $s_t$, let $a_t \in [0, \bar{a}]$ be the search effort for the unemployed agent at time $t$. Then the job arrival intensity is $q_s(a_t)$ with $q'_s(a) > 0$, $q''_s(a) \leq 0$. To simplify the computations, I assume the $q_s(a_t)$ is linear in effort:

$$q_s(a_t) = q_{s0} + q_{s1} a_t,$$

with $q_{s0} \geq 0$, $q_{s1} > 0$. In addition, $q_s(a_t)$ is assumed to have the following property:

$$q_s(a_t) \geq q_s(a_t), \forall a_t \in [0, \bar{a}],$$

which is intended to capture the assumption that the job finding rate is higher in booms than in recessions. Last, I assume that an employed worker loses his job with an exogenous separation rate $p_s$ with $p_B < p_R$, that a healthy agent would become disabled with the rate $\lambda_d$, and that the intensity of transiting from state $s$ to $s'$ is $\lambda_s$, $s \in \{B, R\}$. 


2.2 Preferences and Incentive Compatible Contracts

I assume that an insurance agency (“the principal”) provides disability and unemployment insurance to help the worker smooth his consumption. The worker’s employment status is publicly observable. However, the search effort undertaken by the unemployed worker and the health status of that worker are not observable to the insurance agency. In other words, when a worker reports himself as disabled, the agency cannot tell if the agent is disabled or able to work and search for a job but shirking. Based on this assumption that health status is private information for the agent, the agency cannot distinguish healthy workers from disabled workers. Hence, both moral hazard and private information problem arise, as the agency needs to combat these unobservable factors and offer insurance that induces the unemployed workers to exert effort in finding a new position and induces healthy workers not to misreport themselves as disabled.

I will define $j \in \{E, U\}$ as the employed and unemployed status with $E = 0$ and $U = 1$. Also, let $d \in \{H, D\}$ stand for healthy and disabled with $H = 0$ and $D = 1$. Let the worker’s instantaneous utility be $u(c, a)$ if the consumption is $c$ and effort taken is $a$, with $u$ strictly increasing and concave in $c$ and decreasing and convex in $a$. I also assume the worker dies stochastically with rate $\kappa$. Let the subjective discount rate be $\hat{\rho}$ and thus the effective discount rate becomes $\rho = \hat{\rho} + \kappa$.

I can now describe the contracts. A contract consists of a quadruple of processes $(c, a, j, d) = (\{c_t\}_{t=0}^{\infty}, \{a_t\}_{t=0}^{\infty}, \{j_t\}_{t=0}^{\infty}, \{d_t\}_{t=0}^{\infty})$, where $c$ is the consumption process with $c_t$ being the amount of consumption of the worker promised by the agency at time $t$, $a$ is the process of effort level, $j$ is the process of employment status, and $d$ is the process of the reported health status, with $a_t$, $j_t$ and $d_t$ defined in a similar way. I assume $c_t \in [0, \bar{c}]$, $a_t \in [0, \bar{a}]$. The contract is history dependent in the sense that $c_t$ and $a_t$ depend on the entire history of the worker’s employment status ($\{j_t\}_{t=0}^{\infty}$) and worker’s health status ($\{d_t\}_{t=0}^{\infty}$). Invoking the revelation principle, I will focus on truthfully reporting contracts, where the agent reports his/her true health status and takes the recommended effort. Given the contract $(c, a, j, d)$, the worker chooses effort to maximize his lifetime expected utility:

$$\max_{\hat{a} \in A} E[\rho \int_0^{\infty} e^{-\rho t} u(c, \hat{a}) dt],$$

where $E$ is the expectation operation, and $A = [0, \bar{a}]$. A contract is incentive compatible if and only if the worker (i) exerts the recommended search effort and (ii) truthfully reports his/her health status that solves the problem (1).

Let $v(\cdot)$ be the utility for the insurance agency. The objective of the agency is to design the
contract as follows.

$$\max_{(c,a,j,d)} E[-\rho \int_0^\infty r^{-\rho t}v(c_t - 1(\text{if the worker is employed})\omega)dt]$$

such that

$$(c, a, j, d)$$ is incentive compatible

and

$$E[\rho \int_0^\infty e^{-\rho t}u(c, a)dt] \geq W_0,$$

where $W_0$ is the reservation utility of the worker, $1(.)$ is the indicator function. After specifying the incentive compatible contracts, I will show how to derive the solutions in the next section.

3 Optimal Contract

In this section, I will show that optimal contracts can be derived by using the promised utility of the agent as states and controls. Then I will lay out the corresponding Hamiltonian-Jacobi-Bellman equations describing the optimal contracts.

3.1 Incentives and Promised Utility

In order to solve the optimal contracts, it is useful to first define the compensated martingales. I already defined the aggregate state $s_t \in \{B, R\}$, and now I will assign the numerical values as $R = 1$ being the recessions and $B = 0$ being the booms. Also, recall that $j \in \{E, U\}$ are the employed and unemployed statuses, with $E = 0$ and $U = 1$, and $d \in \{H, D\}$ stand for healthy and disabled with $H = 0$ and $D = 1$. Let the associated compensated jump martingales be $m^j_t$, $m^s_t$, and $m^d_t$ governing the jumps between 0 and 1, with $m^j_t$ and $m^s_t$ being observable to the agency while $m^d_t$ is the reported process. The evolution for the processes of the compensated martingales can be written as

$$dm^j_t = (1 - d_t)(- (1 - j_t)[((1 - s_t)p_G + s_t p_B)] + j_t[(1 - s_t)q_R(a_t) + s_t q_R(a_t)])dt + \Delta j_t$$

$$dm^s_t = -(1 - s_t)\lambda_R + s_t \lambda_R]dt + \Delta s_t$$

$$dm^d_t = -(1 - d_t)\lambda_d)dt + \Delta d_t,$$

where $\Delta$ governs when the worker switches states. For example, for a healthy worker ($d = 0$) who is in his unemployment spell ($j = 1$) while the economy is in a boom ($s = 0$), the compensated jump processes are:

$$dm^j_t = q_R(a_t)dt + \Delta j_t$$

$$dm^s_t = -\lambda_R dt + \Delta s_t$$

$$dm^d_t = -\lambda_d dt + \Delta d_t,$$
where the negative term compensates the positive jumps. This makes the processes mean zero martingales.

Now I am ready to consider the incentive compatible contracts. Given a contract \((c, a, j, d)\) and the arbitrary effort process \(\hat{a}, \hat{j}, \text{and } \hat{d}\), I define the promised utility of the worker as

\[
W_t \equiv E[\rho \int_t^\infty e^{-\rho t}u(c, \hat{a})dt], \forall t \in [0, \infty],
\]

which stands for the expected utility of a worker at time \(t\) given the contract \((c, a, j, d)\) but exerting effort \(\hat{a}\), reporting the employment status \(\hat{j}\) and health status \(\hat{d}\). I will first show the result using the martingale representation theorem.

**Proposition 1.** Under a contract \((c, a, j, d)\) and the chosen effort level \(\hat{a}\), the chosen employment status \(\hat{j}\), and the reported health status \(\hat{d}\), there exists three \(\mathbb{F}\)-predictable\(^3\) processes \(g^j_t\), \(g^s_t\), and \(g^d_t\) such that

\[
E[\int_0^\infty e^{-\rho t}g^j_t dt] < \infty, E[\int_0^\infty e^{-\rho t}g^s_t dt] < \infty, \text{ and } E[\int_0^\infty e^{-\rho t}g^d_t dt] < \infty,
\]

and

\[
dW_t = \rho(W_t - u(c_t, \hat{a}_t))dt + \rho g^j_t dm^j_t + \rho g^s_t dm^s_t + \rho g^d_t dm^d_t.
\]

**Proof.** See Appendix A.1.

Next, I consider the conditions that guarantee the incentive compatible contracts.

**Proposition 2.** Given the results in proposition 1, the contract is incentive compatible if and only if the following holds for all \(t\):\[\]

\[
a_t \in \arg \max_{\hat{a}_t} g^j_t q_{st}(\hat{a}_t) + u(c_t, \hat{a}_t)
\]

\[
g^j_t \leq 0
\]

\[
g^d_t \leq 0.
\]

**Proof.** See Appendix A.2.

Proposition 1 is the standard method used in continuous time dynamic contracting literature, which is demonstrated in Sannikov (2008), Williams (2009, 2011, 2015), Li (2012), and Williams and Li (2014). To solve the dynamic programming problem, I first use the appropriate martingales so that the objective function can be rewritten recursively. In proposition 2, I then show how to use the martingales derived from proposition 1 to express incentive compatible constraints. This way, I am able to write the problem recursively and impose the constraints on the incentive

\(^3\mathbb{F}\)-predictable stands for the sigma-algebra that is generated by the process of \(dm^j_t\), \(dm^s_t\), and \(dm^d_t\).
problems. This simplifies the computations because the solutions are reduced to solving the Hamilton-Jacobi-Bellman (HJB) equations.

For the rest of this section, I will show how to derive the HJB equations governing the optimal contracts using the propositions above.

### 3.2 Value Functions and Optimal Contracts

In this section, I will derive the conditions of the value functions for the insurance agency. Defining $V(W,j,s)$ as the value functions for the agency in state $j$ and $s$ with promised utility $W$ delivered to the worker when healthy and $V(W,d)$ as the value function with $W$ delivered to the disabled worker. Before deriving the Hamilton-Jacobi-Bellman equations, let me consider the boundary points of value functions. Those boundary points will serve as the choice set for the HJB equations. Since I will use the promised utility as choices, I will first consider the possible sets given the boundaries of the parameter values such as consumption and effort. Since the ideas and arguments here are similar to the ones in Williams and Li (2014), I explain these details in Appendix A.3.

#### 3.2.1 The Hamilton-Jacobi-Bellman Equations

After deriving the boundary points, I am ready to specify the HJB equations that determine the optimal contracts. First, it is convenient to change the control variables using the promised utilities as variables. Considering the unemployed worker in the state $s$, if $W^j_t$ is used as the worker’s promised utility immediately after the change of the job status, and $W^d_t$ as the worker’s promised utility immediately after the change of the health status, then the incentive compatible constraints become:

$$g^j_t \Delta j_t = \frac{W^j_t - W_t}{\rho}, \quad g^d_t \Delta d_t = \frac{W^d_t - W_t}{\rho}$$

Then I can rewrite the constraints as

$$a^* \in \arg \max_{\hat{a}} u(c, \hat{a}) + \frac{W^j_t - W_t}{\rho} q_s(\hat{a}), \quad W_t \geq W^d_t.$$  

Similarly, considering the workers in state $s$, the constraints become

$$W_t \geq W^d_t, \quad W^j_t \leq W_t \text{ when employed}, \quad W^j_t \geq W^d_t \text{ when unemployed}.$$

Hence, the HJB equations can be specified as follows:

**Proposition 3.** Suppose the value functions $(V(W,j,s), V(W,d))$ exist and the left and right boundaries are derived in Appendix A.3, then the value functions satisfy a system of HJB equa-
\[ \rho V(W, u, s) = \max_{\hat{c} \in [0, \bar{c}], W^j \in [W^{es}_{r}, W^{es}_{l}], W^s \in [W^{es}_{r}, W^{es}_{l}], W^d \in [W^{ed}_{r}, W^{ed}_{l}], W^j \geq W, W^d \leq W} -\rho v(\hat{c}) \]

\[ \rho V(W, e, s) = \max_{\hat{c} \in [0, \bar{c}], W^j \in [W^{es}_{r}, W^{es}_{l}], W^s \in [W^{es}_{r}, W^{es}_{l}], W^d \in [W^{ed}_{r}, W^{ed}_{l}], W^d \leq W} -\rho v(\hat{c} - \omega) \]

\[ \rho V(W, d) = \max_{c \in [0, \bar{c}]} -\rho v(\hat{c}) + \rho V(W, d)[W - u(\hat{c})] \]

Proof. See Appendix A.4.

In Proposition 3, the contracting problems are reduced to solving a system of HJB equations. The HJB equations capture the expected changes of the agency’s value functions. The second term captures the expected cost of providing consumption and recommended efforts, if any. The last three terms capture the jump processes when the agent’s employment status and health status change. Thus, instead of deriving the history dependent contracts, solving HJB equations is enough to characterize the solutions, which is essentially solving a system of Partial Differential Equations. This simplifies solving the problems to a great extent. For the rest of the paper, I will discuss the implications from this system of HJB equations.

4 A Quantitative Example

In this section, I will discuss the quantitative implications of the model. Throughout this section, I will assume that the utility function of the worker takes the following form, and risk-neutral
agency:
\[ u(c, a) = \frac{c^{1-\gamma}}{1 - \gamma} - \frac{a^{1+\phi}}{1 + \phi}, \quad v(c) = c. \]

4.1 The Benchmark Contract and Cost of the Current System

To determine the key endogenous parameters, the properties of the current system need to be captured. Thus, I will first consider a stylized version of the current system, which can be used to calibrate the model and measure the effects of switching to the optimal insurance system. I will call this stylized version, “the benchmark contract,” where a worker receives constant unemployment benefits \( c_B \) for a fixed length of time and constant disability benefits \( c_d \). Furthermore, I assume that the duration of the unemployment benefits is state dependent, wherein the worker can receive \( T_R \) periods in recessions and \( T_B \) periods in booms, with \( T_B < T_R \).

In order to capture the complexities of the disability insurance screening process, several things must be taken into account. The actual DI application process consists of several steps. First, the worker has a medical disability preventing him from working and cannot earn more than what is called a “substantial gainful amount.” Second, the worker has to apply for disability insurance, and it takes at least three to five months before the awards are granted. Therefore, I make the following simplifications. First, the worker has to make the choice to apply for DI. Second, a worker with a disability will be awarded the benefits, but a healthy worker will be accepted with probability \( \pi_d \). Last, upon having been rejected, the worker cannot receive unemployment insurance or apply for DI with the same reason again unless he/she is hit by a health shock or transitioning back to employment. The last assumption is intended to capture the opportunity cost in reality when workers decide whether to apply for DI or not.

The detailed derivations on solutions to the benchmark contract will be discussed in Appendix B. In this appendix, I will show that the utility level of a worker under the benchmark contract can be obtained by solving a system of differential equations (ODEs). To capture workers claiming disability when they are healthy, I will then replace the differential terms on those utilities that fall below the utility when workers become disabled. In addition, based on the status and choices of the workers, I will show how to calculate the benchmark contract’s corresponding cost to the agency.

4.2 Data and Calibration

The model period is taken as one week. First, following in the steps of the existing literature, I will fix a few parameters following the literature. Following Hopenhayn et al. (1997), the risk aversion is set to be \( \gamma = 0.5 \), and the weekly discount rate is set to be \( \rho = 0.001 \), which corresponds to an annual discount rate of 5%. Next, I will set the weekly wage to be \( \omega = 495 \), which corresponds to
the median annual wage of $25,737 in 2007. In addition, I will set the constant in the job finding rate to be $q_{s0} = 10^{-5}$, which prevents some singularity problems but has no impact on the main results. Also, I will set the maximum consumption that can be allocated to a worker as equal to wage: $\bar{c} = \omega$. This means that the choice set of the consumption is $[0, 2\omega]$. As for the health shock, I will set it as $\lambda_d = 0.0008$, the rate for which is taken from Low and Pistaferri (2015). The acceptance rate for the healthy worker is $0.5$, which is taken from Kitao (2014).

For the benchmark contract, I will set $T_B = 26$ weeks in booms, which is the average duration across U.S. states, and $T_R = 39$ in recessions, which corresponds to the regular federal extended unemployment benefits program. The replacement ratios for unemployed workers will be set to be $c^b = 0.47\omega$, which is consistent with the 47% average replacement ratio in U.S. in 2009, and the replacement ratio for workers with disability is set to be 33%, which is the average replacement ratio for a 40-year old worker in 2007 with median wage.

I will estimate a two-state Markov-switching process using the data from Shimer (2012) to obtain the parameters governing the Markov process for aggregate states and the corresponding job finding and separation rates. This data set contains the quarterly averages of monthly job findings and separation rates from 1948Q1 to 2007Q1. I will focus on the job finding rates, as Shimer emphasized that the cyclicality in the data comes primarily from the job finding rates. In order to focus on the cyclicality components, I will first use the Hodrick-Prescott filter to remove
the low-frequency trend from the job finding rate data. Then, I will estimate the two-state Markov-switching model, following the approach of Hamilton (1989). That is, the H-P filtered job finding rates ($f_t$) are estimated by

$$f_t = m_{st} + \epsilon_t.$$ 

From the estimation, I find that the mean job finding rates in booms and recessions are $m_B = 0.4875$ and $m_R = 0.4107$, with the transition rates 0.9332 and 0.9107. This gives the aggregate economic switching rates as $\lambda_B = 0.0058$ and $\lambda_R = 0.0078$. In addition, the estimated recession indicator, which is when the smoothed probability of a recession is greater than 0.5, is shown in Figure 2. I read the mean job separation rates in booms and recessions from the H-P filtered data, and this gives $p_B = 0.0086$ and $p_R = 0.0089$.

Last, I calibrate the rest of the parameters by simulating a population of 50,000 workers and computing the average job finding rates as well as the elasticities of unemployment duration with respect to an increase in the unemployment benefits. In this simulation, I assume workers start at age 40 and will work 25 years until they retire at age 65. For the elasticities, the typical range of estimates is between 0.5 and 1 (Landais et al. (2013), Chetty (2008)), and I will target the value at the middle of the rate at 0.7. This gives $q_{B1} = 0.0037$, $q_{R1} = 0.0032$ and the effort cost function parameter $\phi = 0.145$.

### 4.3 Quantitative Implications

In this section, I demonstrate the behavioral responses of the optimal contracts. I will particularly focus on comparing the differences between the benchmark system and the optimal contract, focusing especially on the cost reductions from adopting the optimal contracts. In addition, I consider the impacts of different policy reforms.

#### 4.3.1 Cost Reductions from the Optimal System

I will first show the summary statistics from simulations. In Table 1, I list the unemployment rates, unemployment durations, finding rates, and separation rates. As for the data fit, the simulations from the benchmark contract match the data quite well, but the unemployment durations are different between the simulations and the data. The main reason for this is that I match the moments of job finding rates in booms and recessions, and this pins down the unemployment durations because there is no heterogeneity between workers. This problem can be fixed if different cohorts of workers are simulated. As in the optimal system, the unemployment rate drops around

---

4The solutions of the benchmark model is similar to the optimal case. We solve a system of ordinary differential equations (ODEs). Details are presented in the Appendix.
40% and unemployed durations drop more than 50% as workers search harder in the optimal system. I will leave the discussions on the properties of the optimal contracts to Section 5 because they are qualitatively similar, which we obtain deeper intuitions by understanding the implementations of the optimal contracts.

Next, I will demonstrate the cost reductions for the government of switching from the current system to the optimal system. The comparison is calculated by constructing the optimal system that yields the worker the same ex-ante utility. In other words, the worker will be indifferent between switching to the optimal system or staying at the current system. Hence, additional costs/gains only come from the information problem.

Since I assume that only the agency has the techniques to transfer resources across time and that the worker cannot save or borrow, the potential cost savings may be considered as an upper bound on savings because self-insured actions are not allowed in the current system.\(^5\)

In Table 2, the potential gains for the agency are substantial, around 140%.\(^6\) The agency actually collects benefits from the workers. In order to better understand where the cost savings come from, I break down the cost reductions by states. As in the optimal case, the workers pay high taxes when employed and receive higher disability benefits while receiving lower unemployment benefits in the optimal contracts than they do under the benchmark contract. This implies that the incentive problems in the benchmark contract can be handled in a different way without making the workers worse off, but the potential cost reductions to the agency for adopting the optimal contract can still be substantial. In addition, the cost savings in this system come mainly from taxing the employed workers, and there are more people working in the optimal system.

\(^5\)Considering the alternative benchmark model where individual is able to save makes the benchmark case become a system of Differential-Algebraic equations (DAEs). Hence, we will focus on the upper-bound saving instead by solving a system of ordinary differential equations (ODEs).

\(^6\)Table 2 shows zero revenue/expenditure from employed workers. It implies that no additional taxes are collected.
### Government Expenditure (per week per worker)

<table>
<thead>
<tr>
<th>Avg Workers</th>
<th>Benchmark</th>
<th>Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>73.3</td>
<td>-29.9</td>
</tr>
<tr>
<td>Savings (Cost Reductions)</td>
<td>-</td>
<td>140 %</td>
</tr>
</tbody>
</table>

By State

| Unemp (B) | 231.8 | 154.4 |
| Unemp (R) | 232.4 | 234.3 |
| Emp (B)   | 0     | -282.7|
| Emp (R)   | 0     | -228.7|
| Disabled  | 173.4 | 288.2 |

Table 2: Cost Comparisons Between the Benchmark and the Optimal Contract, Wage = 495/week.

#### 4.4 Impact of Different Policy Reforms

In this section, I will consider the impact of different policy reforms on the incentive problems. The first exercise is motivated by the extended UI benefits during the Great Recession, when workers could receive up to 99 weeks of unemployment insurance. The second exercise is intended to show the interdependence of unemployment insurance and disability insurance. As explained above, the existence of disability insurance provides workers with extra search incentives, and the second exercise shows how much their search efforts can be induced using disability insurance alone.

For the first exercise, I will simulate the benchmark model with $T_R = 99$. This means that workers in recessions can receive unemployment insurance for up to 99 weeks. I will compare these results to the case where $T_R = 39$, as workers can receive up to 39 weeks of unemployment insurance benefits in recessions. These results are shown in Table 3, where we can observe that the differences are small in terms of unemployment rate and government expenditure between the system with 39 weeks of unemployment insurance benefits in recessions and the system with 99 weeks of UI in recessions. Although workers search slightly harder in recessions under the system of 39 weeks, most workers find a job before they exhaust their unemployment insurance. This result is similar to Rothstein (2011). I further show that the government expenditure is similar between the two systems. This implies that extending the unemployment insurance duration does not help to solve the incentive problems.

Last, I will consider the second exercise: the government takes the current unemployment insurance system as a given but reconsiders the design of the optimal disability insurance. In this exercise, I assume that the government cannot change the unemployment insurance benefits $c^b$ and the maximum unemployment insurance durations $T_B = 26, T_R = 39$. However, the government
can reconsider the design of optimal disability insurance. Through this exercise, we can show the interdependence of the two insurance programs as well as how much search efforts can be motivated by potential future disability insurance benefits.
In Table 4, the unemployment rate drops also around 40% and the cost reductions are around 40% if the government switches from the current system to the system that redesigns the DI program. This means that a big proportion of the search effort problem due to the incentives in the current system can be solved even if only the optimal disability insurance is implemented. This counterfactual indicates that DI is being used as an additional tool for the government to provide individuals with incentives to search for jobs. When the joint design is considered, the driving forces behind the substantial cost reductions of the optimal system come from the availability of this additional tool. In the next section, I confirm that these savings result from the implementation of the optimal contracts and show the robustness of this implication.

5 Implementation of the Optimal Contracts

In this section, I consider a special case that allows me to analyze the implementation of the optimal contracts. Using the model that implements the optimal contracts allows me to obtain deeper insights regarding the substantial cost savings possible by considering disability insurance and unemployment insurance together. To achieve this goal, I will first assume the utility functions to be exponential. The solutions of the optimal contracts can be reduced to a system of non-linear algebraic equations, which is much easier to solve when compared to Partial Differential Equations. Next, I will show that the optimal contracts under exponential utilities can be implemented by a worker's consumption-saving-effort model, as in Williams and Li (2014). Making this simplification helps us to understand the properties of the optimal contracts by observing the worker's borrowing and saving activities. The solvable and implementation case, in particular, help us to gain more insight when the joint insurance design is considered, which is difficult to infer by only observing the solutions of the HJB equations.

The arguments in this section follow closely the ones demonstrated in Williams and Li (2014). In Williams and Li (2014), the authors show that using the exponential utilities and shutting down the channel of job separation yield solvable solutions and that the optimal contracts can be implemented by allowing workers to save or borrow via different interest rates plus flow payments and lump-sum transfers. In this paper, I extend Li and Williams’ work by showing that their methodology can be extended without shutting down the possibility of getting separated from jobs, and that optimal contracts are implementable through a similar worker’s consumption-saving-effort model.

I will focus mainly on how the implementation case can help us to understand the results from the previous quantitative section. All the calculations will be provided in Appendix C, and the complete comparative analysis will be provided in the online appendix for reasons of space.
5.1 A Solvable Special Case

Throughout this section, I assume the workers’ preferences are given by:

\[ u(c, a) = \exp(-\theta_A(c - h(a))) \]

where \( h \) is increasing and convex with \( h(0) = 0 \) and \( \theta_A > 0 \) is the risk aversion coefficient. The agent’s cost function is given by:

\[ v(c) = \exp(\theta_P c) \]

where \( \theta_P > 0 \) is the risk aversion coefficient. By making this assumption, I show that solutions of the optimal contracts take the following forms:

\[
\begin{align*}
V(W, e, s) &= -V_e^*(s)(-W)^{-\frac{\theta_P}{\theta_A}}, c(W, e, s) = \log V_e^*(s) \frac{1}{\theta_P + \theta_A} \log(-W) + \frac{\theta_P}{\theta_P + \theta_A} \omega \\
v(W, u, s) &= -V_u^*(s)(-W)^{-\frac{\theta_P}{\theta_A}}, c(W, u, s) = \log V_u^*(s) \frac{1}{\theta_P + \theta_A} \log(-W) + \frac{\theta_A}{\theta_P + \theta_A} h(a^*(s)) \\
v(W, d) &= -(W)^{-\frac{\theta_P}{\theta_A}}, c(W, d) = \frac{1}{\theta_A} \log(-W),
\end{align*}
\]

where \( a^*(s) \) is the effort level in private information case, \( c(.) \) is the consumption, and \( V_u^*(s), V_e^*(s) \) are constants depending on the aggregate economic condition \( s \). The eight unknowns \{\( a^*(s), V_u^*(s), c^*(s), V_e^*(s), s \in \{B, R\} \}\) can be obtained by solving a system of eight non-linear equations presented in Appendix C.

5.2 A Worker’s Consumption-Savings-Effort Problem

Upon obtaining the solvable special case, I will show how the optimal contracts under exponential utilities can be implemented via a rather simple model: a worker’s consumption-savings-effort model. In particular, the instruments are (1) allowing workers to save or borrow using a bond and (2) providing flow payments and lump-sum transfers (or payments) where the interest rates and the amounts paid (transferred) depend on the employment or health status of the agent and the state of the economy. This allows me to gain insight into the properties of the optimal contracts, where the behavior of promised utility can be explained by a worker’s self-insured actions. As explained in the previous section, the ideas behind this model come from Li and Williams [2014], where they consider the implementation of optimal UI with business cycles. The main difference is that I need to show how to pin down the indeterminacy of the flow payments, which depend on the state from which the workers transit. Again, detailed calculations will be provided in the Appendix C.

To be specific, I consider an environment where a worker has wealth \( x_t \) and has access to a bond with an instantaneous rate of return \( r^d \) when disabled, \( r^e(s_t) \) when employed, and \( r^u(s_t) \)
when unemployed. In addition, this worker will receive a constant $b^d$ when disabled, $b^e(s_t)$ when employed, and $b^u(s_t)$ when unemployed. Third, an employed worker receives a lump-sum transfer $B^e_d(s_t)$ if he becomes disabled, $B^e_u(s_t)$ if he becomes unemployed, and $A^e(s_t, x_t)$ when the aggregate economy state switches. An unemployed worker receives $B^u_d(s_t)$ when he becomes disabled, $B^u_e(s_t)$ when he finds a job, and $A^u(s_t, x_t)$ when the aggregate economy state switches.

The idea of implementation is that we allow the workers to self-insure, and the flow payments plus lump-sum transfers induce the correct incentives for the worker to search for a new job and not apply for disability if healthy. Hence, the worker needs to decide how much to save (borrow) and consume and the effort level they are willing to put forth in each period. In Appendix C, I will show the conditions when the consumption-savings-effort model implements the optimal contracts. In particular, I will show how the unknown parameters that govern the solutions can be pinned down.

For the rest of this section, I will focus on demonstrating the theoretical properties of the optimal contract. In particular, I will show how substantial cost reductions can be achieved under the optimal contracts when the joint design of the insurance system is considered.

5.3 Analysis

For the rest of this section, I will illustrate the dynamics of the optimal contracts under different information structures and aggregate economic conditions, using the the worker’s consumption-savings-effort model. I will use the parameter values from the calibration results in the later section: $q_B = 0.0037$, $q_R = 0.0032$, $\lambda_B = 0.0058$, $\lambda_R = 0.0078$, $\rho = 0.001$, and $\omega = 495$. In addition, I choose $\theta_A = 0.0015$, $\theta_P = 0.0005$, and $h(a) = \nu a^{1+\phi}/(1+\phi)$ with $\nu = 0.01$ and $\phi = 1.7$ for illustration purposes. I will focus on what we can learn from the implementation of the optimal contracts and demonstrate only the selected comparative statics. The comparative statics of the solvable case will be presented in the online appendix.

In Figure 3, I plot the effective interest rate when unemployed and employed. The effective interest rate is greater than the subjective discount rate ($r^u(s) > \rho$) when unemployed, but smaller than the subjective discount rate $r^e(s) < \rho$ when employed, so the contract provides an interest rate subsidy to the unemployed workers and taxes the employed workers. The subsidy or tax increase when the workers are more risk averse, but the subsidy also increases while the tax decreases when the agency is more risk averse. The subsidy increases when it is easier to find a job, and the tax increases when it is easier to be separated from a job. Not surprisingly, the subsidy and the tax increase when it is easier for the workers to get hit by a health shock.

In Figure 4, I plot the lump sum payments $B^u_d(s)$ when the worker finds a job and the lump sum payment $B^u_e(s)$ when the worker gets separated from a job. The lump sum payments of
Figure 3: Comparative Statics of Effective Interest Rate when Unemployed $r^u(s)$ and Employed $r^e(s)$.

Figure 4: Comparative Statics of Lump-Sum Transfers when a Worker’s Employment Status Changes. $B^u_e(s)$ are positive and larger in recessions than in booms so as to provide incentives. The number also does not drop a lot as the probability of people who become disabled increases. In addition, the lump sum transfer $B^u_0(s)$ when the worker loses his job is negative, reflecting the fact that the worker is punished when he becomes separated from the job. Also, the lump sum transfers when the worker finds a new job are higher than when the worker loses his original job. This fact induces the worker to search even harder for a job, as he can then accumulate more wealth. This observation also explains the fact that workers with smaller average unemployment durations
receive higher disability insurance benefits, as disability insurance provides extra incentives for the workers to search harder for jobs.

In Figure 5, I plot the lump sum transfers when the worker becomes disabled from being unemployed $B^d_{t}(s)$ and employed $B^e_{t}(s)$. The lump sum transfers when a worker becomes disabled are negative, meaning that the optimal contracts induce people to truthfully report by lowering the promised utility. This negative amount is substantially larger when the probability of becoming disabled is lower. Also, a worker has to pay more when he transits from employed to disabled, reflecting the fact that this transition is unfavorable to the agency in the optimal system.

5.3.1 Implications

The main purpose behind this section is to explain the driving forces behind the optimal joint insurance design and what causes the substantial cost reductions in the previous section by discussing the implications from the implementation of the optimal contracts. The main conclusion of this study is that disability insurance can serve as an additional tool for the government to provide incentives for job searching when the joint insurance design and the overlap between disability insurance and unemployment insurance are considered. Hence, the substantial cost savings of the optimal system proposed here can only be achieved when disability insurance and unemployment insurance are considered jointly rather than cases where the government takes only unemployment insurance or disability insurance into account. This implication can be easily observed from the implementation of the optimal contracts.

Let us look at the interest rate first. Since $r^e(s) < \rho < r^u(s)$, it implies that the government
taxes the employed while providing subsidies for unemployed workers in the optimal contracts. The intuition behind this is that the purpose of joining the insurance program is to get insured from the consumption uncertainty, and that giving unemployed workers a higher interest rate allows them to self-insure and provides incentives for them to find a job more quickly. This explains how the incentives are provided in the optimal system.

Next, let me look at the values of $B_u(s)$, $B_e(s)$. The first thing we can observe is that $|B_u(s)| > |B_e(s)|$, the absolute value of lump-sum transfer when an unemployed worker finds a job, is higher than the absolute value of the lump-sum transfer when an employed worker loses a job. This implies that strong incentives are provided to search for jobs in the optimal contracts. In other words, the worker’s consumption increases when finding a job, but decreases when losing a job. In addition, a worker with a shorter average unemployment duration enjoys higher compensations in the optimal contracts, because by finding a job more quickly, his wealth increases more ($|B_u(s)| > |B_e(s)|$) than it would if he were to stay unemployed. To be more specific, each time a worker finds a job, he gets compensated around 3000 dollars. At the same time, each time a worker loses a job, he gets punished by around 1200 dollars. This means that a worker transiting from employed to unemployed and back to employed receives around 1800 net compensation. This explains how the incentives are given in the optimal contracts.

Last, let us consider the values of $B_u(s)$, $B_e(s)$. The values are all negative. This implies that workers are punished when applying for disability insurance. It also implies that the workers’ consumption decrease when the workers apply for disability insurance in the optimal contracts. This is not surprising because that is how the incentives are provided. In addition, the smaller value of $B_u(s)$ than $B_e(s)$ implies that the punishment is higher for an unemployed worker applying for disability insurance under the optimal system. This shows that incentive problems are stronger when a worker is unemployed and immediately shows that disability insurance in the optimal contracts serves as an additional tool for the government to provide incentives for job searching, because in the model that implements the optimal contracts, the compensation punishes workers who claim disability if they do not search for a job hard enough before the application.

In sum, we show how substantial cost savings can be achieved by considering a rather simple model that implements the optimal system. Different from the case when the government considers only the unemployment insurance or disability insurance, the government has two tools (DI and UI) that can be used together to provide incentives for searching for jobs. This explains why the substantial savings in the quantitative section can be obtained only if the joint design is considered.
6 Conclusion

In this paper, I consider the optimal joint insurance system against disability and unemployment shocks with cyclical fluctuations. I analyze how the optimal contracts should be designed given the information problem where the government faces workers potentially overstating their health status and how hard they are searching for a job. I then consider a stylized version of the current system and evaluate the cost reductions when switching to the optimal system. I find that the cost reductions from the incentive problems are substantial, while the extended UI benefits, as given in the recent recession, has no impacts on these incentive problems. In addition, disability insurance can be used as an additional tool for the government to provide incentives for workers searching for jobs. Last, in order to understand the properties of the optimal contract, I consider the implementation of the optimal contracts through saving and borrowing behavior with state-switching payments. The implementation helps to understand the driving forces behind the optimal design through the worker’s borrowing or saving behavior. The results indicate clearly through what channels disability insurance can serve as a policy tool to provide incentives.

Simplifying the model helps to focus mainly on the incentive problems when the joint insurance design is considered. It allows me to isolate the cyclicality feature of the DI and the connections between unemployment insurance and disability insurance. However, there are limitations to this model. The current model is not able to address some important issues existing in the current structure of DI and UI and their impact on labor force participation. This model simplifies the DI application by assuming that workers can be either healthy or with disability, while disability is an absorbing state. In addition, jobs are assumed to be homogenous and unemployed workers stay in the labor force even after their benefits expire. Although the assumptions help to solve for tractable results and simplify the computations, the problems posed by the mismatch of jobs, the decisions that workers with mild disability will have to make, and the decision to stay in the labor force remain unanswered. All of these issues are important and my model could be extended to address these issues for future studies.

Appendix

A Proofs

A.1 Proof for Proposition 1

Proof. Define

\[ \phi_t(c,a) = E_t[\rho \int_0^\infty e^{-\rho s} u(c_s, a_s)ds | \mathbb{F}_t] = \rho \int_0^t e^{-\rho s} u(c_s, a_s)ds + e^{-\rho t} W_t. \]
Since \( u(c, a) \) is bounded for the bounded sets of \( c \) and \( a \). Also, \( \phi_t(c, a) \) is uniform integrable. By the martingale representation theorem, there exist three \( \mathbb{F}_t \)-predictable square integrable processes \( g_t^j, g_t^s, g_t^d \) such that

\[
d\phi_t(c, a) = \beta e^{-\rho t} g_t^j dm_t^j + \beta e^{-\rho t} g_t^s dm_t^s + \beta e^{-\rho t} g_t^d dm_t^d.
\]

Combining the equations gives the result.

\[
\Box
\]

A.2 Proof for Proposition 2

Proof. The incentive problems come from three sources: the worker’s effort is unobservable, employed workers can voluntarily quit their jobs, and healthy people can misreport as disabled. Since the incentive problems are state-contingent, i.e. unemployed workers cannot quit their jobs since they are not employed, I will prove this proposition state by state. I will start by showing the necessity first and then the sufficiency.

Let us first consider the case when the worker is unemployed. Given a contract \((c, a, j = 1, d = 0)\), I define \( \phi_t(c, a') \) for some feasible \( a' \) and health status \( d' \) as

\[
\phi_t(c, a') = \rho \int_0^t e^{-\rho u(c_s, a'_s)} ds + e^{-\rho W_t}, \forall t \in [0, \infty)
\]

It is clear that for any alternative \((a', d')\), \( \phi_0(c, a') = W_0 \). Differentiating the previous equation with respect to \( t \) gives:

\[
d\phi_t(c, a') = \beta e^{-\rho t} u(c_t, a'_t) dt - \rho e^{-\rho t} dW_t
\]

\[
= -\rho e^{-\rho t} [u(c_t, a'_t) - c(c_t, a_t)] dt + g_t^j dm_t^j + g_t^s dm_t^s + g_t^d dm_t^d.
\]

Let \( dm_t^j \) be the compensated jump martingale with \( a' \) and \( dm_t^d \) be the martingale associated with \( d' \). Then

\[
dm_t^j = [(1 - s_t)q_B(a'_t) + s_t q_R(a'_t)] - [(1 - s_t)q_B(a_t) + s_t q_R(a_t)] dt + dm_t^j
\]

\[
dm_t^d = \lambda_d dt + dm_t^d.
\]

Hence,

\[
d\phi_t(c, a') = \beta e^{-\rho t} [u(c_t, a'_t) - c(c_t, a_t)] + g_t^j [(1 - s_t)q_B(a'_t) + s_t q_R(a'_t)]
\]

\[
- [(1 - s_t)q_B(a_t) + s_t q_R(a_t)] dt + g_t^j dm_t^j + g_t^s dm_t^s + g_t^d dm_t^d.
\]

Under \( a', d' \), \( dm_t^j \), \( dm_t^d \) are martingales. Then, the drift term of \( \phi_t(c, a') \) has the same sign as

\[
[u(c_t, a'_t) - c(c_t, a_t)] + g_t^j [(1 - s_t)q_B(a'_t) + s_t q_R(a'_t)] - [(1 - s_t)q_B(a_t) + s_t q_R(a_t)].
\]

Thus, if

\[
a_t \in \arg \max_{a'_t} g_t^j q_{a_t}(a'_t) + u(c_t, a'_t)
\]

\[
g_t^d \leq 0,
\]

\( \phi_t(c, a') \) is a sub-martingale, so I have

\[
E[a', d'][\phi_t(c, a')] > \phi_0(c, a') = W_0,
\]

which implies that \((a', d')\) dominates \((a, d)\), hence it is not optimal. The case when the worker is employed can be proved in a similar way. Hence I prove the necessity.

For sufficiency, suppose \((c, a, j, d)\) satisfies the conditions. Then, \( \phi_t(c, a') \) is a super-martingale for any other alternative feasible choices. Since \( c_t, a_t, d_t, j_t \) are all bounded, I have

\[
W_0 = \phi_1(c, a') \geq E[\phi_\infty(c, a')].
\]

Hence \((c, a, j, d)\) dominate all other feasible choices.

\[
\Box
\]
A.3 The Boundary Points of the Value Functions

I will show how the left and right boundaries are derived in this appendix. The arguments follow closely those in Williams and Li (2014).

First, the left boundaries are computed by considering the harshest contract that the principal can offer to the agent: this contract will give the agent the minimum level of consumption while forcing the agent to choose the highest effort level. The right boundaries are computed by solving the most generous contract that the agency can offer: giving the highest level of consumption to the agent. I will use \((W_{ij}^{js}, V(W_{ij}^{js}, j, s))\), \((W_{r}^{d}, V(W_{r}^{d}, d))\) for the right boundaries for the worker and the agency with status \(j \) or \(d\) and state \(s\). The left boundaries are denoted as \((W_{ij}^{js}, V(W_{ij}^{js}, j, s))\), \((W_{r}^{d}, V(W_{r}^{d}, d))\). Then I will have the following proposition:

**Proposition 4.** The bounds of the promised utility for a worker are \(W_{ij}^{js} = W_{ij}^{d} = \underline{w}, \) for \(j = E, U, \) and \(s = B, R\). The corresponding value functions for the insurance agency \(V(W_{ij}^{js}, j, s), \) and \(V(W_{r}^{d}, d)\) are

\[
\rho V(W_{ij}^{us}, u, s) = \lambda_s (V(W_{ij}^{us'}, u, s') - V(W_{ij}^{us}, u, s)) + q_d (V(W_{ij}^{us}, e, s) - V(W_{ij}^{us}, u, s)) \\
+ \lambda_d (V(W_{ij}^{d}, d) - V(W_{ij}^{us}, u, s)) \\
\rho V(W_{ij}^{es}, e, s) = \rho c(v) + \lambda_s (V(W_{ij}^{es'}, e, s') - V(W_{ij}^{es}, e, s)) + p_s (V(W_{ij}^{us}, u, s) - V(W_{ij}^{es}, e, s)) \\
+ \lambda_d (V(W_{ij}^{d}, d) - V(W_{ij}^{es}, e, s)) \\
\rho V(W_{ij}^{d}, d) = 0.
\]

The upper bounds of the promised utility for the worker \(W_{ij}^{js}, W_{r}^{d}\) satisfy the following:

\[
\rho W_{ij}^{us} = \max_{\alpha \in [0, \tilde{\alpha}]} \rho u(\tilde{\alpha}) + \lambda_s (V(W_{ij}^{us'}, u, s') - V(W_{ij}^{us}, u, s)) + q_d (V(W_{ij}^{us}, e, s) - V(W_{ij}^{us}, u, s)) + \lambda_d (W_{ij}^{d} - W_{ij}^{us}) \\
\rho W_{ij}^{es} = \rho u(\tilde{\alpha}_s) + \lambda_s (W_{ij}^{es'} - W_{ij}^{es}) + p_s (W_{ij}^{us} - W_{ij}^{es}) + \lambda_d (W_{ij}^{d} - W_{ij}^{es}) \\
\rho W_{ij}^{d} = \rho u(\tilde{\alpha}),
\]

with \(V(W_{ij}^{js}, j, s) = V(W_{ij}^{d}, d) = -\rho v(\tilde{\alpha}).\)

**Proof.** Let us start with left boundaries. Since all the proofs are similar, let us focus on the case when the worker is unemployed and in state \(B\). The other left boundaries can be proven in a similar way.

Let \(\tau = \tau^j \land \tau^s \land \tau^d\), where \(\tau^j\) is the time when the next job arrives, \(\tau^s\) is the time when the next aggregate state changes, and \(\tau^d\) is the time when the next health shock hits. For any \(\Delta > 0\), since consumption is zero, I have

\[
V(W_{ij}^{us}, u, s) = e^{-\rho(\Delta \land \tau)} (P(\tau > \Delta) V(W_{ij}^{us}, u, s)) + e^{-\rho(\Delta \land \tau)} [P(\tau < \Delta, \tau = \tau^j) V(W_{ij}^{es}, e, s) + P(\tau < \Delta, \tau = \tau^d) V(W_{ij}^{d}, d)].
\]

Subtracting \(V(W_{ij}^{us}, u, s)\) and dividing \(\Delta\) on both sides, I have

\[
0 = e^{-\rho(\Delta \land \tau)} e^{-(\lambda_s + q_d + \lambda_d)\Delta [V(W_{ij}^{us'}, u, s') - V(W_{ij}^{us}, u, s)]} \\
+ e^{-\rho(\Delta \land \tau)} (1 - e^{-(\lambda_s + q_d + \lambda_d)\Delta}) \frac{q_d}{\lambda_s + q_d + \lambda_d} V(W_{ij}^{es}, e, s) \\
+ \frac{\lambda_s}{\lambda_s + q_d + \lambda_d} V(W_{ij}^{es'}, e, s') + \frac{\lambda_d}{\lambda_s + q_d + \lambda_d} V(W_{ij}^{d}, d).\]

Taking \(\Delta \to 0\), I have the result.

Next, I consider the right boundaries. Let us consider the case of the unemployed worker in state \(s\). All the other cases can be proven in a similar way.
The unemployed worker in state $s$ faces the following problem:

$$
\max_{a \in A} E_t [\rho \int_{t}^{t+\Delta} e^{-\rho t} u(\bar{c}, a) dt + e^{-\rho (s + \lambda_s (a_t) + \lambda_d) \Delta W_{rs}^u} + (1 - e^{-\lambda_s + q_s(a_t) + \lambda_d}) \frac{\lambda_s}{\lambda_s + q_s(a_t) + \lambda_d} W_{rs}^u + \frac{q_s(a_t)}{\lambda_s + q_s(a_t) + \lambda_d} W_{rs}^u + \frac{\lambda_d}{\lambda_s + q_s(a_t) + \lambda_d} W_{rs}^d].
$$

The solution to the problem is $W_{rs}^u$. Hence for any $a \in A$, I have:

$$
W_{rs}^u \geq \max_{a \in A} E_t [\rho \int_{t}^{t+\Delta} e^{-\rho t} u(\bar{c}, a) dt + e^{-\rho (s + \lambda_s (a_t) + \lambda_d) \Delta W_{rs}^u} + (1 - e^{-\lambda_s + q_s(a_t) + \lambda_d}) \frac{\lambda_s}{\lambda_s + q_s(a_t) + \lambda_d} W_{rs}^u + \frac{q_s(a_t)}{\lambda_s + q_s(a_t) + \lambda_d} W_{rs}^u + \frac{\lambda_d}{\lambda_s + q_s(a_t) + \lambda_d} W_{rs}^d].
$$

Subtracting $V(W_{rs}^u, u, s)$, dividing $\Delta$ on both sides, and taking $\Delta \to 0$, I have the result.

**A.4 Proof for Proposition 3**

\textbf{Proof.} I will show the HJB equation when the economy is in a boom period and the worker is unemployed. Other HJB equations can be proven in a similar manner.

Assuming that at time $t$, the promised utility of the worker is $W_t = W$. Let $c$ be the consumption process, $W^j$ be the adjusted utility when the employment status changes, $W^u$ be the adjusted utility when the aggregate state changes, and $W^d$ be the adjusted utility when the health shock hits. Let $\tau^j$, $\tau^u$, and $\tau^d$ be the stopping time when the next job, aggregate state changes, and health shock arrive, respectively. Defining $\tau = \tau^j \wedge \tau^u \wedge \tau^d$. At time $t$ and for small interval of time $\Delta$, I have

$$
V(W_t, u, B) \geq E_t [\rho \int_{t}^{t+\Delta} e^{-\rho t} u(c_s) ds | F_{\tau^j} -] + e^{-\rho (t+\Delta) \wedge \tau} \Pr(\tau > t + \Delta | a)V(W_{t+\Delta}, u, B) + e^{-\rho (t+\Delta) \wedge \tau} \Pr(\tau > t + \Delta, \tau = \tau^j | a)V(W_{t+\Delta}^j, e, B) + e^{-\rho (t+\Delta) \wedge \tau} \Pr(\tau > t + \Delta, \tau = \tau^u | a)V(W_{t+\Delta}^u, u, R) + e^{-\rho (t+\Delta) \wedge \tau} \Pr(\tau > t + \Delta, \tau = \tau^d | a)V(W_{t+\Delta}^d, d).
$$

Let $\Delta$ be small enough such that everything is well-defined and effort is constant in the interval. Subtracting $V(W_t, u, B)$ on both sides and dividing both sides by $\Delta$, I have

$$
0 \geq -\frac{1}{\Delta} E_t [\rho \int_{t}^{t+\Delta} e^{-\rho t} u(c_s) ds | F_{\tau^j} -] + 1 \Delta e^{-\rho (t+\Delta) \wedge \tau} \frac{q_{a}^j(W_{t+\Delta}^j, W_{t+\Delta}^j)}{\lambda_B + q_B(a^*(W_{t+\Delta}^j, W_{t+\Delta}^j)) + \lambda_d} V(W_{t+\Delta}, e, B) + e^{-\rho (t+\Delta) \wedge \tau} \frac{q_{a}^u(W_{t+\Delta}^u, W_{t+\Delta}^u)}{\lambda_B + q_B(a^*(W_{t+\Delta}^u, W_{t+\Delta}^u)) + \lambda_d} V(W_{t+\Delta}, u, R) + e^{-\rho (t+\Delta) \wedge \tau} \frac{q_{a}^d(W_{t+\Delta}^d, W_{t+\Delta}^d)}{\lambda_B + q_B(a^*(W_{t+\Delta}^d, W_{t+\Delta}^d)) + \lambda_d} V(W_{t+\Delta}, d).
$$

Let $\Delta \to 0$, and I have

$$
0 \geq -\rho(u(c_t)) + V(W_t, u, B) + V_R(W_t, u, B) \rho |W_t - u(c_t, a_t) - q_B(a(W_t^j, W_t^j))| \frac{W_t^j - W_t}{\rho} - \lambda_B W_t^u - W_t - \lambda_d W_t^d - \frac{\lambda_d}{\rho}
$$

$$
+ q_B(a(W_t^j, W_t^j)) [V(W, e, B) - V(W, u, B)] + \lambda_s [V(W^u, u, R) - V(W, u, B)] + \lambda_d [V(W^d, d) - V(W, u, s)].
$$

Since $c_t$, $a_t$, $W_t^j$, $W_t^u$, $W_t^d$ are chosen optimally, the inequality holds with equality. \qed
B Benchmark Contract

I will show how to calculate the promised utility for the worker under the benchmark contract. Because of the assumption of the absorbing state, it is easier to consider the case without misreporting and then add the choice of misreporting back into the calculations subsequently.

Let me begin by considering the case when misreporting is not allowed. First, I assume that the expected utility for the worker in state $s$ under the benchmark contract is $W^*_s$, and that the expected utility for a disabled worker is $W^d$. I will assume the utility is given, and I will show that solving the benchmark model is equivalent to solving a system of ordinary differential equations. Next, I consider the period after $T_R$, in which the unemployed worker receives zero unemployment benefits. Let $\tau = \tau^j \wedge \tau^s \wedge \tau^d$, where $\tau^s$ is the first date of a switch in aggregate state, $\tau^j$ is the first time that this worker finds a job, and $\tau^d$ is the first time that the disability shock arrives. I also denote the utility in this region by $W^{u3}_s$. Following similar steps as I did in deriving the HJB equation above, I can see that, for $t > T_R$ the values of $W^{u3}_s$ are

$$\rho W^{u3}_s = \max_{a \in [0, a]} \left[ \rho(u(0) - h(a)) + q_s(a)(W^*_s - W^{u3}_s) + \lambda_e(W^e_s - W^{u3}_s) + \lambda_d(W^d - W^{u3}_s) \right].$$

Then, let us consider time in $[T_B, T_R]$, where the worker only receives benefits in a recession. I will denote the utility of the worker in state $s$ during this period as $W^{u2}_s(t)$. Similar to the previous case, the pair of HJB equations is as follows:

$$\rho W^{u2}_R(t) - \frac{d}{dt} W^{u2}_R(t) = \max_{a \in [0, a]} \left[ \rho(u(c^B) - h(a)) + q_R(a)(W^e_s - W^{u2}_R(t)) + \lambda_e(W^e_R(t) - W^{u2}_R(t)) \right] + \lambda_d(W^d - W^{u2}_R(t))$$

$$\rho W^{u2}_B(t) - \frac{d}{dt} W^{u2}_B(t) = \max_{a \in [0, a]} \left[ \rho(u(0) - h(a)) + q_B(a)(W^e_s - W^{u2}_B(t)) + \lambda_e(W^e_B(t) - W^{u2}_B(t)) \right] + \lambda_d(W^d - W^{u2}_B(t))$$

with boundary conditions $W^{u2}_R(T_R) = W^{u3}_R$ and $W^{u2}_B(T_R) = W^{u3}_R$. Last, in the region of $[0, T_R]$, the promised utility, which I denote by $W^{u1}_s(t)$, evolves as:

$$\rho W^{u1}_s(t) - \frac{d}{dt} W^{u1}_s(t) = \max_{a \in [0, a]} \left[ \rho(u(c^B) - h(a)) + q_s(a)(W^*_s - W^{u1}_s(t)) + \lambda_e(W^e_s(t) - W^{u1}_s(t)) \right] + \lambda_d(W^d - W^{u1}_s(t))$$

with boundary conditions $W^{u1}_s(T_R) = W^{u2}_s(T_B)$.

Adding back in the choice of misreporting is straightforward, since I assume that disability is an absorbing state. In the promised utility solved in the equations above, I know that the agent would misreport if $W^*_s(t) < W^d$. As a consequence, $\frac{d}{dt} W^*_s(t) = 0$. In sum, the set of HJB and ordinary differential equations are

$$\rho W^e_s = \begin{cases} \rho u(\omega) + p_s(W^*_s(0) - W^*_s) + \lambda_e(W^e_s - W^*_s) + \lambda_d(W^d - W^*_s), & \text{if } W^*_s \geq W^d \\ \rho u(0) - h(a) + q_s(a)(W^e_s - W^{u3}_s) + \lambda_e(W^e_s - W^{u3}_s), & \text{otherwise} \end{cases}$$

$$\rho W^{u3}_s = \begin{cases} \rho u(0) - h(a) + q_s(a)(W^e_s - W^{u3}_s) + \lambda_e(W^e_s - W^{u3}_s), & \text{if } W^{u3}_s \geq W^d \\ \rho u(0) - h(a) + q_s(a)(W^e_s - W^{u3}_s) + \lambda_e(W^e_s - W^{u3}_s), & \text{otherwise} \end{cases}$$

$$W^d = u(c^d).$$
let me verify the guess for the disabled workers first. The first order condition with respect to the consumption
for the disabled worker takes the form of:

\[ \rho W^{u_d}_R(t) - \frac{d}{dt} W^{u_d}_R(t) = \begin{cases} 
\max_{a \in [0, \bar{a}]} \left\{ \rho(a(c_R^B) - h(a)) + q_R(a)(W^{u_d}_R(t)) - \rho W^{u_d}_R(t) - W^{u_d}_R(t) + \lambda_d(W^d - W^{u_d}_R(t)), \right. \\
\left. \text{if } W^{u_d}_s \geq \text{report disabled,} \right. \\
\text{otherwise} 
\end{cases} \]

Next, let me consider the case for employed workers. Using the conjectured form, the slope matching conditions

\[ \frac{\partial}{\partial \theta} \left[ \frac{\rho W^{u_d}_B(t) - \frac{d}{dt} W^{u_d}_B(t)}{\rho W^d_B(t)} \right] = \frac{\rho W^{u_d}_B(t) - \frac{d}{dt} W^{u_d}_B(t)}{\rho W^d_B(t)} + \lambda_d(W^d - W^{u_d}_B(t)), \]

if \( W^{u_d}_s \geq \text{report disabled} \)

\[ \frac{\partial}{\partial \theta} \left[ \frac{\rho W^{u_d}_B(t) - \frac{d}{dt} W^{u_d}_B(t)}{\rho W^d_B(t)} \right] = \frac{\rho W^{u_d}_B(t) - \frac{d}{dt} W^{u_d}_B(t)}{\rho W^d_B(t)} + \lambda_d(W^d - W^{u_d}_B(t)), \]

if \( W^{u_d}_s \geq \text{report disabled} \)

\[ \frac{\partial}{\partial \theta} \left[ \frac{\rho W^{u_d}_B(t) - \frac{d}{dt} W^{u_d}_B(t)}{\rho W^d_B(t)} \right] = \frac{\rho W^{u_d}_B(t) - \frac{d}{dt} W^{u_d}_B(t)}{\rho W^d_B(t)} + \lambda_d(W^d - W^{u_d}_B(t)), \]

if \( W^{u_d}_s \geq \text{report disabled} \)

\[ \frac{\partial}{\partial \theta} \left[ \frac{\rho W^{u_d}_B(t) - \frac{d}{dt} W^{u_d}_B(t)}{\rho W^d_B(t)} \right] = \frac{\rho W^{u_d}_B(t) - \frac{d}{dt} W^{u_d}_B(t)}{\rho W^d_B(t)} + \lambda_d(W^d - W^{u_d}_B(t)), \]

if \( W^{u_d}_s \geq \text{report disabled} \)

C Calculations for the Solvable Case and the Implementation

C.1 Calculations for the Solvable Case

Let me first consider the solvable case. I assume the forms of the value function for the agency are:

\[ V(W, d) = -(-W)^{\frac{\theta_p}{\theta_A}}, \quad V(W, e, s) = -V^*_e(s)(-W)^{\frac{\theta_p}{\theta_A}}, \quad V(W, u, s) = -V^*_u(s)(-W)^{-\frac{\theta_p}{\theta_A}}. \]

The goal is to show a system of equations that jointly determine the parameters.

Let me verify the guess for the disabled workers first. The first order condition with respect to the consumption
for the disabled worker gives

\[ -\theta_P \exp(\theta_P \cdot c) = \frac{\theta_P}{\theta_A} (-W)^{\frac{\theta_p}{\theta_A} + \theta_A} \exp(-\theta_A c) \Rightarrow c = -\frac{1}{\theta_A} \log(-W) \]

Plugging everything back into the HJB equation, I can see that all terms cancel out, which verifies that the original
guess is correct.

Next, let me consider the case for employed workers. Using the conjectured form, the slope matching conditions gives

\[ W^j = \left( \frac{V^*_e(s)}{V^*_u(s)} \right)^{-\frac{\theta_p}{\theta_A}} W, \quad W^s = \left( \frac{V^*_e(s)}{V^*_u(s)} \right)^{-\frac{\theta_p}{\theta_A} + \theta_A} W, \quad W^d = V^*_u(s)^{-\frac{\theta_p}{\theta_A}} W. \]

Hence, I have the following:

\[ V(W^j, u, s) - V(W, e, s) = \left[ -V^*_e(s)^{\frac{\theta_p}{\theta_A} + \theta_A} V^*_u(s)^{\frac{\theta_p}{\theta_A}} + V^*_e(s)(-W)^{\frac{\theta_p}{\theta_A}} \right] (W^j)^{-\frac{\theta_p}{\theta_A}} \]

\[ V(W^s, e, s) - V(W, e, s) = \left[ -V^*_e(s)^{\frac{\theta_p}{\theta_A} + \theta_A} V^*_u(s)^{\frac{\theta_p}{\theta_A}} + V^*_e(s)(-W)^{\frac{\theta_p}{\theta_A}} \right] (W^s)^{-\frac{\theta_p}{\theta_A}} \]

\[ V(W^d, d) - V(W, e, s) = \left[ -V^*_e(s)^{\frac{\theta_p}{\theta_A} + \theta_A} + V^*_e(s)(-W)^{\frac{\theta_p}{\theta_A}} \right] (W^d)^{-\frac{\theta_p}{\theta_A}}. \]

Next, the optimality conditions for \( c \) gives:

\[ c = \frac{1}{\theta_p + \theta_A} \log(V^*_e(s)) - \frac{1}{\theta_A} \log(-W) + \frac{\theta_A}{\theta_p + \theta_A} \omega \]

Using the optimality condition for \( c \) and slope matching conditions, plugging them into the HJB equation, and
canceling the \( W \) term gives

\[ V^*_e(s)^{-\frac{\theta_p}{\theta_A}} \left[ \rho \exp(\frac{\theta_p \theta_A}{\theta_p + \theta_A} \omega) + p_s V^*_u(s)^{-\frac{\theta_p}{\theta_A} + \theta_A} + \lambda_s V^*_e(s)^{-\frac{\theta_p}{\theta_A} + \theta_A} + \lambda_d \right] = \left( \rho + p_s + \lambda_s + \lambda_d \right). \]

Last, let me consider the case for the unemployed workers. I further assume the consumption for the unemployed
worker takes the form of:

\[ c(W, u, s) = c^*(s) + h(a^*(s)) - \frac{1}{\theta_A} \log(-W). \]
Using the conjectured forms, I have the following results:

\[ v(c) = \exp(\theta_P(c^*(s) + h(a^*(s))))(-W)^{-\frac{\theta_P}{\rho}} \]

\[ u(c, a^*(s)) = -\exp(-\theta_A c^*(s))(-W) \]

\[ u_e(c, a^*(s)) = \theta_A \exp(-\theta_A c^*(s))(-W) \]

\[ u_u(c, a^*(s)) = -\theta_A h'(a^*(s))\exp(-\theta_A c^*(s))(-W) \]

\[ u_{uu}(c, a^*(s)) = -[\theta_A^2 h''(a^*(s))] + \theta_A h''(a^*(s))\exp(-\theta_A c^*(s))(-W) \]

\[ u_{uu}(c, a^*(s)) = \theta_A^3 h'(a^*(s))\exp(-\theta_A c^*(s))(-W) \]

Next, the incentive compatible constraint for the effort gives: \(-u_a(c, a) = q_s \frac{W^u - W}{\rho} \). Using the optimality conditions for \( c \) and \( a \), as well as the HJB equation, straightforward but tedious calculations yield three additional non-linear equations below:

\[ \exp((\theta_P + \theta_A)c^*(s) + \theta_P h(a^*(s))) = V_u^*(s)(1 - a^*(s)\theta_A h'(a^*(s))) \]

\[ + \frac{\theta_A}{\rho} V_u^*(s)(1 - \frac{\theta_A}{\rho} \theta_A h'(a^*(s))\exp(-\theta_A c^*(s))) - \frac{\theta_A^3 h'(a^*(s))\exp(-\theta_A c^*(s))}{\theta_A^2} \]

(2)

\[ a^*(s)\theta_P h'(a^*(s)) - h''(a^*(s))(s)\exp(-\theta_A c^*(s))(1 - \frac{\theta_A}{\rho} \theta_A h'(a^*(s))\exp(-\theta_A c^*(s))) - \frac{\theta_A^3 h'(a^*(s))\exp(-\theta_A c^*(s))}{\theta_A^2} \]

(3)

\[ 0 = (\rho + q_s a^*(s) + \lambda_d)V_u^*(s) - \rho \exp((\theta_P + \theta_A)c^*(s)) + \frac{\theta_P}{\theta_A} V_u^*(s)(1 - \exp(-\theta_A c^*(s))) \]

\[ + a\theta_A \exp(-\theta_A c^*(s))h'(a^*(s)) - \frac{\lambda_a}{\rho} ((V_u^*(s) - \frac{\theta_A}{\rho} V_u^*(s)) - 1) - \frac{\lambda_d}{\rho} ((V_u^*(s) - \frac{\theta_A}{\rho} V_u^*(s)) - 1) \]

\[ - q_s a^*(s)V_u^*(s)(1 - \frac{\theta_A}{\rho} \theta_A h'(a^*(s))\exp(-\theta_A c^*(s))) - \frac{\theta_A^3 h'(a^*(s))\exp(-\theta_A c^*(s))}{\theta_A^2} \]

\[ - \lambda_a V_u^*(s') \frac{\theta_A}{\rho} \theta_A V_u^*(s) - \frac{\theta_A}{\rho} \theta_A V_u^*(s') - \lambda_d V_u^*(s) \frac{\theta_A}{\rho} \theta_A . \]

(4)

Thus, the eight unknowns \{\( a^*(s), V_u^*(s), c^*(s), V_e^*(s), s \in \{B, R\} \} \} can be solved by the system of eight non-linear equations: (1), (2), (3), (4).

C.2 Calculations for the Implementation Case

From the model setup, the problem for the workers’ consumption-savings-effort model is characterized by the following three HJB equations:

\[ \rho J(x, d) = \max \rho u(c) + J_d(x, d)(r^d x - c + b^d) \]

\[ \rho J(x, e, s) = \max \rho u(c) + J_e(x, e, s)[r^e(s)x + c + b^e(s)] + p_u[J(x, s)] \]

\[ + \lambda_s[J(x + s^*(x), e, s') - J(x, e, s)] + \lambda_d[J(x + B^d(s), d) - J(x, e, s)], \]

\[ \rho J(x, u, s) = \max \rho u(c) + J_u(x, u, s)[r^u(s)x + c + b^u(s)] + q_a[J(x + B^u(s), e, s) - J(x, u, s)] \]

\[ + \lambda_s[J(x + A^u(s), e, s') - J(x, u, s)] + \lambda_d[J(x + B^u(s), d) - J(x, u, s)], \]

where \( J(x, d), J(x, e, s), J(x, u, s) \) are the value functions for disabled workers, employed workers in state \( s \) and unemployed workers in state \( s \) respectively. In this section, I will guess that the solutions to the value functions take the following form:

\[ J(x, d) = e^{-\theta_A r^d x}, \quad J(x, e, s) = -J_u(s)e^{-\theta_A r^u(s)x}, \quad J(x, u, s) = -J_u(s)e^{-\theta_A r^u(s)x} \]

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The goal is to verify that these guesses are correct and to show the conditions when this consumption-savings-effort model implements the optimal contracts. To be more specific, I show how (1) the deposit interest rate: \( r^d \), \( r^e(s) \), and \( r^u(s) \), (2) the flow payments: \( b^d \), \( b^e(s) \), and \( b^u(s) \), and (3) the lump-sum transfers: \( B^e_0(s) \), \( B^e_0(s) \), \( B^e_0(s) \), \( B^e_0(s) \), \( A^e(s, x) \), and \( A^u(s, x) \) are determined. The difficulty comes from the indeterminacy of some parameters that have to be determined jointly.

Let me consider the case for disabled workers. The optimality condition for \( c \) gives

\[
c(x, d) = -\frac{1}{\theta_A} \log(r^d J_d) + r^d x.
\]

Substituting everything back to the HJB equation yields:

\[
J(x, d) = -J_d \exp(-r^d \theta_A x), \text{ where } J_d = \frac{\rho}{r^d} \exp(-\frac{\rho - r^d \theta_A x b^d}{r^d}).
\]

Also, the consumption becomes

\[
c(x, d) = \frac{\rho - r^d}{\theta_A r^d} + b^d + r^d x.
\]

As this model implements the optimal contracts, the promised utility must equal the value function. Hence,

\[
W = J(x, d) = \frac{\rho}{r^d} \exp(-\frac{\rho - r^d \theta_A x b^d}{r^d}).
\]

Thus, the consumption in the solvable case then becomes

\[
c(W(x), d) = \frac{1}{\theta_A} \log(J_d) + r^d x.
\]

In addition, the consumption must be equal in the optimal contracts and in this model: \( c(W(x), d) = c(x, d) \). Thus, I obtain that

\[
r^d = \rho, c(x, d) = b^d + \rho x, \quad J(x, d) = -\exp(-\theta_A (b^d + \rho x)).
\]

We can observe that \( b^d \) and \( x \) cannot be determined at this stage. I will resolve this problem when I solve the problem for the employed and unemployed worker in the later subsections.

For the problem of an employed worker, I again first assume the constant amount \( b^d \) that the disabled worker can receive if he transitions from unemployed status depends on his/her wealth, the current aggregate state, and the difference in the interest rate. Let \( b^d = (r^e(s) - \rho)x_T - \frac{\rho}{r^d} \log J_e(s) \), where \( x_T \) is the wealth of this unemployed worker at time \( T \) when a disability shock hits this worker. In addition, \( b^u(s) \) will make the value function for the unemployed workers have the following form:

\[
J(x, u, s) = -J_e(s) \exp(-\theta_A r^e(s)x_T).
\]

The optimality condition for \( c \) becomes \( \rho u_t(c) = J_e(x, e, s) \). I further assume the \( A^e(s, x) \) takes the form of

\[
A^e(s, x) = \left( \frac{r^e(s)}{r^e(s')} - 1 \right) x + \frac{r^e(s)}{r^e(s')} \hat{A}^e(s).
\]

Following the guess, the consumption then becomes

\[
c(x, e, s) = -\frac{1}{\theta_A} \log(r^e(s) \rho) - \frac{1}{\theta_A} \log(J_e(s)) + r^e(s)x.
\]

Using the guesses for the value function, I have

\[
\begin{align*}
J(x + B^e_0(s), u, s) - J(x, c, s) &= (1 - \exp(-\theta_A r^e(s) B^e_0(s))) J_e(s) \exp(-\theta_A r^e(s)x) \\
J(x + B^e_0(s), d) - J(x, c, s) &= (1 - \exp(-\theta_A r^e(s) B^e_0(s))) J_e(s) \exp(-\theta_A r^e(s)x) \\
J(x + A^e(s, x), u, s') - J(x, c, s) &= (1 - \exp(-\theta_A r^e(s) \hat{A}^e(s))) J_e(s) \exp(-\theta_A r^e(s)x).
\end{align*}
\]
To implement the optimal contracts, I first set \( W = J(x, e, s) \). Thus, the consumption in the optimal contracts becomes
\[
c(W(x), e, s) = \frac{\log V_e(s)}{\theta_A + \theta_P} + \frac{\theta_P}{\theta_A + \theta_P} \omega - \frac{1}{\theta_A} \log(-W)
\]
To make the consumption equal: \( c(x, e, s) = c(W(x), e, s) \), I must have
\[
\frac{r^e(s)}{\rho} = \exp(-\frac{\theta_A \theta_P}{\theta_A + \theta_P} \omega)V_e(s) - \frac{\theta_A}{\theta_A + \theta_P}
\]
which pins down \( r^e(s) \).

For the other variables, recalling that
\[
x_t = -\frac{\log(W_t) + \log J_e(s)}{r^e(s)\theta_A} = X_t + \log J_e(s) - \frac{r^e(s)}{\theta_A}.
\]
By the generalized Itô’s lemma, I can derive the evolution of the wealth:
\[
dx_t = -\mu_W(s_t) dt - \log(w^e(s_t)) \Delta s^e_t - \left( \frac{\log(w^e(s_t))}{r^e(s_t)\theta_A} + (1 - \frac{r^e(s)}{r^e(s')} x_t) \right) \Delta s^e_t - \log(w^e(s_t)) \Delta s^d_t.
\]
However, the evolution for this consumption-savings-effort problem is
\[
dx_t = \frac{1}{\theta_A} \log\frac{r^e(s_t)}{\rho} + \frac{1}{\theta_A} \log J_e(s) + b^e(s_t) dt + B^e_u(s_t) \Delta s^e_t + A^e(s, x) \Delta s^e_t + B^e_d(s) \Delta s^d_t.
\]
Hence I can pin down \( b^e(s), B^e_u(s), B^e_d(s), \) and \( A^e(s) \) as
\[
\begin{align*}
b^e(s_t) &= -\frac{1}{\theta_A} \log\frac{r^e(s_t)}{\rho} - \frac{\mu_W(s_t)}{r^e(s_t)\theta_A} - \frac{1}{\theta_A} \log J_e(s) \\
B^e_u(s_t) &= -\frac{\log(w^e(s_t))}{r^e(s_t)\theta_A} \\
A^e(s_t) &= -\frac{\log(w^e(s_t))}{r^e(s_t)\theta_A} \\
B^e_d(s_t) &= -\frac{\log(w^e(s_t))}{r^e(s_t)\theta_A}
\end{align*}
\]
Next, I will verify that the guess of the value function solves the implementation problem. Using the previous results, and canceling the terms in \( x \), the HJB equation becomes:
\[
0 = \rho - r^e(s) + \theta_A r^e(s) \left( \frac{1}{\theta_A} \log\frac{r^e(s)}{\rho} + \frac{1}{\theta_A} \log J_e(s) + b^e(s) \right) + p_u[1 - \exp(-\theta_A r^e(s) B^e_u(s))] + \lambda_a[1 - \exp(-\theta_A r^e(s) A^e(s))] + \lambda_d[1 - \exp(-\theta_A r^e(s) B^e_d(s))].
\]
Substituting back the policies for \( b^e(s) \) (up to a scale of \( J_e(s) \), \( B^e_u(s), B^e_d(s), \) \( A^e(s) \) into the HJB equation above allows me to verify that all terms cancel out, which verifies my guess and shows that the policy implements the optimal contract. The indeterminacy of \( b^e(s), b^u(s), J^e(s) \) will be discussed in the last part.

Next, let me consider the problem of the worker. I again first assume the constant amount \( b^d \) that the disabled worker can get if he transits from unemployed status depends on his/her wealth, the current aggregate state, and the difference in the interest rate. Let \( b^d = (r^u(s_t) - \rho) x_T - \frac{1}{\theta_A} \log J_u(s) \), where \( x_T \) is the wealth of this unemployed worker at time \( T \) when a disability shock hits this worker. In addition, \( b^e(s) \) will make the value function for the employed workers have the following form:
\[
J(x, e, s) = -J_u(s) \exp(-\theta_A r^u(s)x_T).
\]
The optimality conditions for \( e \) and \( a \) are
\[
\rho u_e(c, a) = J_u(x, u, s), \quad -\rho u_a(c, a) = q_a[J(x_t + B^u(s_t), e) - J(x, u, s)].
\]
I further assume the $A^u(s, x)$ takes the form of $A^u(s, x) = \frac{(x^u(s) - \kappa_s x)}{r^u(s)} - 1)x + \frac{\kappa^u(s)}{r^u(s)A^u(s)}$. Following the guess, the consumption then becomes

$$c(x, u, s) = -\frac{1}{\theta_A} \log(\frac{r^u(s)}{\rho}) - \frac{1}{\theta_A} \log(J_u(s)) + r^u(s)x + h(a).$$

Using the guesses for the value function, I have

$$J(x + B^u(s), c, s) - J(x, u, s) = (1 - \exp(-\theta_A r^u(s)B^u(s)))J_u(s)\exp(-\theta_A r^u(s)x)$$

$$J(x + B^u(s), d) - J(x, u, s) = (1 - \exp(-\theta_A r^u(s)B^u(s)))J_u(s)\exp(-\theta_A r^u(s)x)$$

$$J(x + A^u(s, x), u, s) - J(x, u, s) = [1 - \exp(-\theta_A r^u(s)\hat{A}^u(s))]J_u(s)\exp(-\theta_A r^u(s)x).$$

The optimality condition for $a$ becomes

$$\rho \theta_A h'(a) \exp(-\theta_A(c - h(a))) = q_a(1 - \exp(-\theta_A r^u(s)B^u(s)))J_u(s)\exp(-\theta_A r^u(s)x),$$

which the terms in $c$ can be canceled. This verifies the optimal $a = a^*(s)$ is independent of $x$. To implement the optimal contracts, I first set $W = J(x, u, s)$. Thus, the consumption in the optimal contracts becomes

$$c(W(x), u, s) = c^*(s) + h(a^*(s)) + r^u(s)x.$$

To make the consumption equal: $c(x, u, s) = c(W(x), u, s)$, I must have

$$c^*(s) = -\frac{1}{\theta_A} \log(\frac{r^u(s)}{\rho}),$$

which pins down $r^u(s)$. For the other variables, recalling that

$$x_t = -\frac{\log(W_t) + \log(J_u(s))}{r^u(s)\theta_A} = \frac{X_t + \log(J_u(s))}{r^u(s)\theta_A}.$$ 

By the generalized Ito's lemma, I can derive the evolution of the wealth:

$$dx_t = \mu^u(W_t)dt - \log(w^u(s))\Delta s_t^d - \left(\frac{\log(w^u(s))}{r^u(s)\theta_A} + 1 - \frac{r^u(s)}{r^u(s')}x_t\right)\Delta s_t^s - \log(w^u(s))\Delta s_t^d.$$ 

However, the evolution for this consumption-savings-effort problem is

$$dx_t = \frac{1}{\theta_A} \log(\frac{r^u(s)}{\rho}) + \frac{1}{\theta_A} \log(J_u(s)) - h(a^*(s)) + b^u(s_t)dt + B^u(s_t)\Delta s_t^d + A^u(s, x)\Delta s_t^s + B^u_d(s_t)\Delta s_t^d.$$ 

Hence I can pin down $b^u(s)$, $B^u(s_t), B^u_d(s_t)$, and $\hat{A}(s_t)$ as

$$b^u(s_t) = -\frac{1}{\theta_A} \log(\frac{r^u(s_t)}{\rho}) - \frac{\mu^u(s_t)}{r^u(s_t)\theta_A} - \frac{1}{\theta_A} \log(J_u(s)) + h(a^*(s))$$

$$B^u(s_t) = \frac{\log(w^u(s_t))}{r^u(s_t)\theta_A}$$

$$\hat{A}(s_t) = \frac{\log(w^u(s_t))}{r^u(s_t)\theta_A}$$

$$B^u_d(s_t) = \frac{\log(w^u(s_t))}{r^u(s_t)\theta_A}.$$ 

Next, I will verify that the guess of the value function solves the implementation problem. Using the previous results, and canceling the terms in $x$, the HJB equation becomes:

$$0 = \rho - r^u(s) + \theta_A r^u(s) \left(\frac{1}{\theta_A} \log(\frac{r^u(s)}{\rho}) + \frac{1}{\theta_A} \log(J_u(s)) - h(a(s)) + b^u(s) + q_a(s)[1 - \exp(-\theta_A r^u(s)B^u(s))] + \lambda_s[1 - \exp(-\theta_A r^u(s)\hat{A}^u(s))] + \lambda_d(1 - \exp(-\theta_A r^u(s)B^u_d(s))).$$
Substituting back the policies for $b^u(s)$ (up to a scale of $J_u(s)$), $B^u_2$, $B(s)$, $\hat{A}(s)$ into the HJB equation above allows me to verify that all terms cancel out, which verifies our guess and shows that the policy implements the optimal contract. Last, it remains to determine $b^e(s), b^u(s), J_e(s), J_u(s)$, which can be determined by the following:

$$-\theta_A b^e(s') - \log\left(\frac{r^e(s')}{\rho}\right) - \mu_{W}(s') = \log J_e(s),$$

$$\theta_A((r^e(s') - r^u(s))x_T - b^u(s')) - \frac{\mu_{W}(s)}{r^u(s)} - \log\left(\frac{r^u(s)}{\rho}\right) = \log J_u(s),$$

$$-\theta_A b^u(s') - \log\left(\frac{r^u(s')}{\rho}\right) - \mu_{W}(s') + \theta_A h(a^*(s')) = \log J_u(s),$$

$$\theta_A((r^u(s) - r^e(s))x_T - b^e(s)) - \frac{\mu_{W}(s')}{r^u(s)} - \log\left(\frac{r^e(s)}{\rho}\right) = \log J_u(s).$$

This completes the calculations of the implementation of the optimal contracts.

References


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